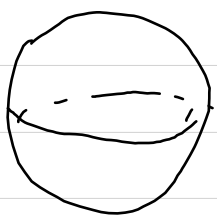
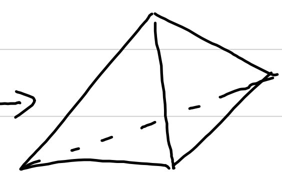
Discrete and Computational Topology

Aim and plan:

The study of discrete/discritised topologic objects.
and the algorithm processing.

e.g. how to detect the existence of some kinds of shapes.

Ex.  S^2 surface, $\|x\|=1$ $\longrightarrow$  a tetuatetion.

discretization of S^2

Computation of Euler

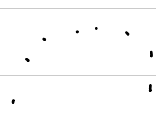
$\chi(S^2) = \chi(\triangle) = 4 - 6 + 4 = 2$ topological

$\underbrace{4}_{\#\text{ vertices}} - \underbrace{6}_{\#\text{edges}} + \underbrace{4}_{\#\text{faces}}$

$\chi(\square) = 8 - 12 + 6 = 2$

Ex. Given some sensor network. (e.g. smoke detection. intusion detection system)
how "good" is the network?

a)  evenly placed network

b) 

How can we dectect "holes" in the networks, to place extra sensors.

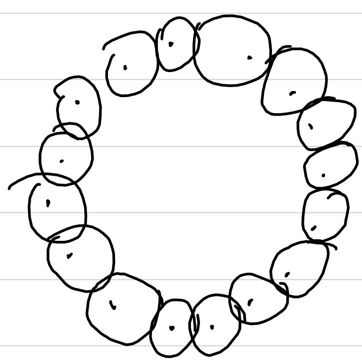
→ Use a topological procedure:

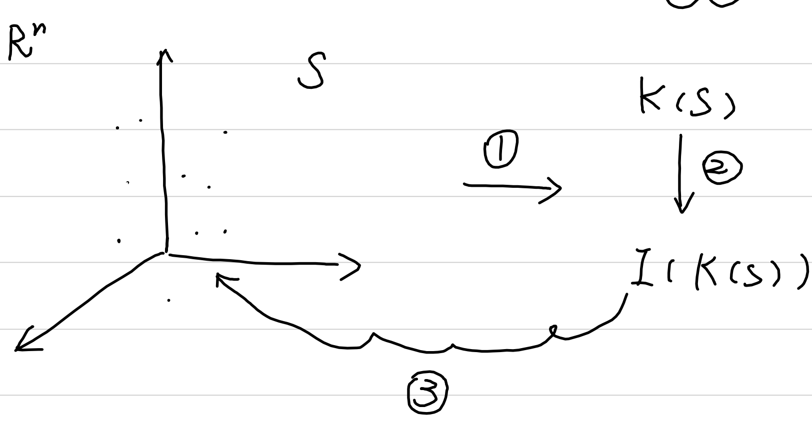
Input: Some (finite) data set $S \subseteq R^n$

① Associate to S a topological space $K(S)$

② Compute topological invariants $I(K(S))$
(e.g. homology on the fundamental group) of $K(S)$

③ use the results of ②
for the further anolysis/understanding/interpolation of the data set S.





Remarks: ① There is a choice of different possible types of complexes for $K(S)$ along different ways.

Associate $K(S)$ with S.

② we need invariants that

- △ are (generally) computable.
- △ can be computed efficiently
- △ are meaningful.

E.g. △ homology

- △ percistent homology
- [△ fundametal group]

$$G_1 = < x \mid x^2 = 1 > = \mathbb{Z}_2$$

$$G_1 = < x, y \mid x = y,\ y^2 = 1 >$$

③ We used visualizations of the results of ② of the initial data set S.

Our input set S can be geometric data of points in R^2 or R^3.
or also data in some very high-dimensional space.

E.g. German data in cancer researsal.

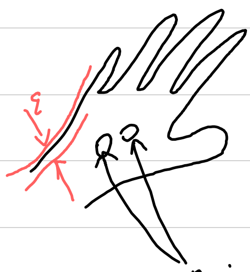
→ libraries are available.

→ Aim: Find correlations

- △ If there are effects at genes A and B ⇒ high cancer risk,
- △ Harder to predict if more genes involved.

Ex: Data set S from 2D or 3D scan

e.g. by placing a hand on a scanner.



noise.

Image processing:

Aim: Reduce noise!

Q: How can we distinguish between noise and feature?

One approach:

Persistent homology.

Compute high-dimensional homology groups H_i^{ε} for different choices of ε.

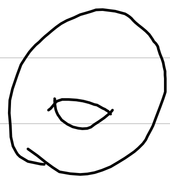
$\rightarrow$ This allows to detect holes that go away quickly (noise) and holes that persist (feature)

Applications of persistent homology / topological data analysis (TDA)

- sensor network.
- gene data
- finance data.
- porosity of materials (e.g. in catalogues)
- ....

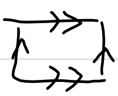
Back to discretizations of spaces / topological objects?

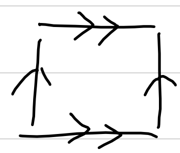
Aim: Decomposition of a top. object into limitedly many pieces that can be handled on a computer.

Ex:  torus T^2 torus does not have vertices or edges, only one face.

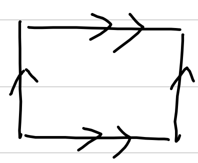
w' different discrezations

Δ as a CW complex (to be defined later)

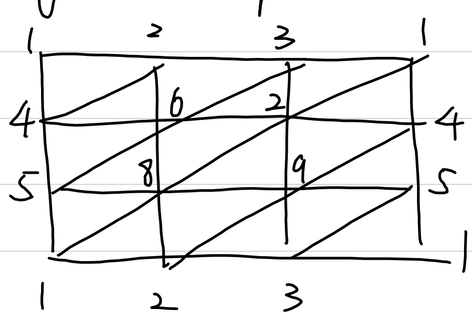
$\chi($  $) = 1 - 2 + 1 = 0$



Δ as a Δ complex (every piece is simplex)



Δ as a (finite) simplicial complex.



Week1 Lecture 2

Q. Can every top object be decomposed (nicely) into finitely many pieces?

A: No in general, not even for compact manifolds?

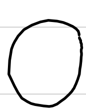
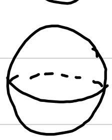
Theorem [Rado 1925, Moise 1952]
Every compact surface (i.e. 2-dimensional manifolds) and every 3-manifold can be triangulated as a finite abstract simplicial complex.

Theorem [Freedman 1982, Paulman 2003]
There are compact 4-manifolds that cannot be triangulated.

Theorem [Manolescu 2013]
For every $d \geq 5$ there are compact d-manifolds that cannot be triangulated
(Disproof of the "triangulation conjecture")

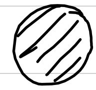
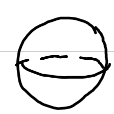
Part I : Discrete Topology
Aim: The study of discretization of topological spaces.

Common objects in topology:

- S^1  circle.
- S^2  2-dimensional sphere
-

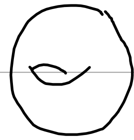
$\vdots$

S^d

- D^1  B^1
- D^2  B^2
- D^3  B^3

$\vdots$

- D^d B^d

△ T^2  torus

2-dim torus

2-

$T^2 = S^1 \times S^1$

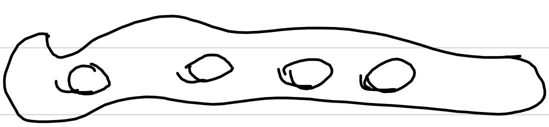
△ T^d d-torus, $T^d = (S^1)^{\times d} = \underbrace{S^1 \times \dots \times S^1}_{d\text{-times}}$

△ orientable surfaces of higher genus

 double torus.

 triple torus.

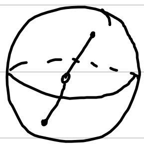
...

 orientable surface of genus g.

g times

- non-orientable surfaces

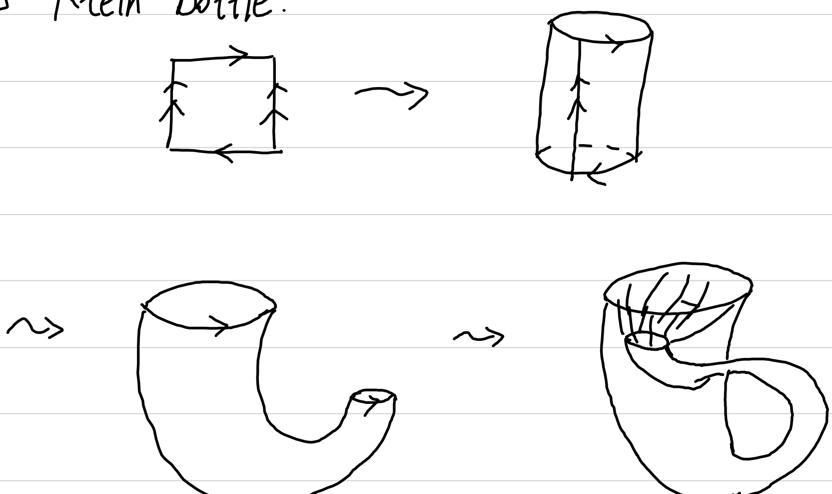
  $\mathbb{RP}^2$ obtained by identifying antipodal points on the 2-sphere.



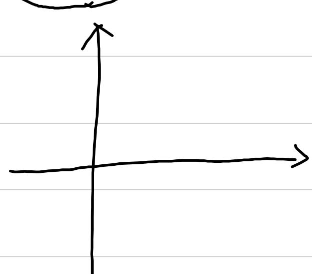
i.e. the upper hemisphere is identifical the lower hemisphere

i.e. as a dist

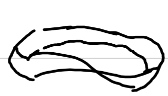
- Klein bottle.



- compact spaces
- Euclidean plane $\mathbb{E}^2$



- higher-dimensional manifolds with or without boundary.
- Mobius band



Ch 1: Metric and topological spaces.

Recall from your calculus or topology class:

Definition (Metric):

Let X be a set
and $d: X \times X \to \mathbb{R}_{>0}$ be a map.
Then d is called a metric
if
i) $d(x,y) = 0 \iff x = y$ (positive definite)
ii) $d(x,y) = d(y,x)$ (symmetry)
iii) $d(x,z) \leq d(x,y) + d(y,z)$. (triangle inequality)
for all $x, y, z \in X$
The pair (X, d) is a metric space.

Ex: $X = \mathbb{R}^d$ $\quad d_E(x,y) = \sqrt{\sum_{i=1}^{d} (x_i - y_i)^2} = \sqrt{\langle x-y, x-y \rangle} = \|x-y\|$

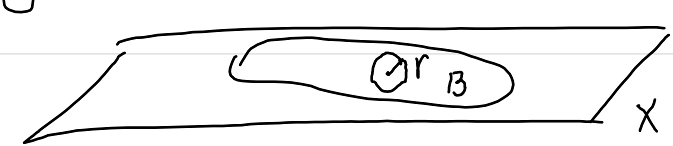
Euclidean metric induced by the standard Euclidean norm $\|\cdot\|_E$

$\mathbb{E}^d = (\mathbb{R}^d, d_E)$

Euclidean d-space.

Def: Let (X, r) be a metric space, $x \in X$, $r \in \mathbb{R}_{>0}$,
then $\mathring{B}_r(x) := \{ y \in X \mid d(x,y) < r \}$
is the open disc resp. open ball center X and radius r.

Def: Let (X, d) be metric space. A subset $O \subseteq X$ is open,
iff every $x \in O$ there is some $r > 0$, s.t $\mathring{B}_r(x) \subseteq O$



Theo An open ball is open.

Ex: Consider $\mathbb{E}^1$

Q: What are the open sets of $\mathbb{E}^1$

A: - open intervals
- unions of open intervals.
- finite intersections of intervals.
- ϕ. $\mathbb{E}^1$

Theorem: Let (X, d) be a metric space.

Then (i) $\bigcup_{i \in I} O_i$ is open for every index set I and O_i open in X.

(ii) $\bigcap_{i \in I} O_i$ is open for any finite intersections.

(iii) ϕ. X are open.

△?

Proof of ii) Let $O = \bigcap_{i=1}^{k} O_i$, O_i open in X, $x \in O$

Then for every $i \in \{1, 2 \dots k\}$

there is some $r_i > 0$.

s.t. $\mathring{B}_{r_i}(x) \subseteq O_i$

Let $r := \min \{r_1, r_2 \dots r_{\xi}\}$. then $r > 0$ and $\mathring{B}_r(x) \subseteq \bigcap_{i=1}^{\xi} O_i$ $\square$

Ex: The intersection of infinitely many open sets needs not to be open.

$$\bigcap_{\xi=1}^{\infty} [x - \tfrac{1}{\xi}, x + \tfrac{1}{\xi}] = x$$

We next generalize metric spaces:

Def (topology)

Let X be a set.

A collection $\mathcal{O} \subseteq 2^X$ of subsets of X is a topology on X.

if i) $\phi, X \in \mathcal{O}$

ii) $\bigcup_{i \in I} O_i \in \mathcal{O}$ for many index set I and $O_i \in \mathcal{O}$

iii) $\bigcap_{i \in I} O_i \in \mathcal{O}$ for many finite index set I and $O_i \in \mathcal{O}$

The subsets $O \subseteq X$ w/ $O \in \mathcal{O}$ are called open, their compoments $X \setminus O$ closed

The pair $(X, \mathcal{O})$ is a topogical space.

Ex: Let X be a set.

- $\mathcal{O} = \{\phi, X\}$ is the indiscrete topology on X
- $\mathcal{O} = 2^X$ is the discrete topology on X.

Week 2 lecture 1

Def: (Continuous map) ✲

Let (X, Θ_X) and (Y, Θ_Y) be topological spaces

A map $f: X \to Y$ is continous if $f^{-1}(O) \in \Theta_X$ for every $O \in \Theta_Y$ open.
i.e. preimages of open set are open.

Example: If we choose $(X, \{\emptyset, X\})$ and $(X, 2^X)$ (in this order)
as topological spaces then $f := id : X \to X$ is not continuous
whereas for the reserved order $id: X \to X$ is continuous. (for $|X| \geq 2$)

Remark: For a top space (X, Θ_X) we often simply write X if Θ_X is understood.

Def: (homeomorphism)

Let X, Y be top spaces, then $f: X \to Y$ is a homemorphism
if f is bijective and f and f^{-1} are continuous maps.
we then say X and Y are homemorphics $X \cong Y$

Def: (neighbourhood) Let (X, Θ) be a topological space and $x \in X$
A subset $N \subseteq X$ is a neighbourhood of x if
there is some $O \in \Theta$ $x \in O \subseteq N$

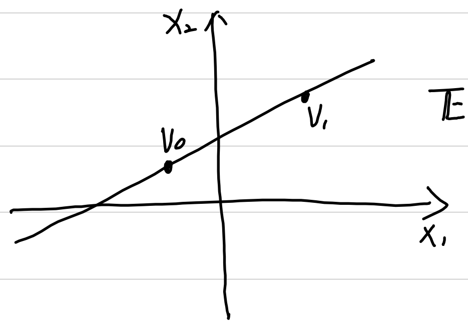
Def (compact) A topological space (X, Θ) is compact if every oper cover
of X has a limit cover. i.e. for $X = \bigcup_{i \in I} O_i$ with $O_i \in \Theta$
there is a finite subfamily $y \in I$ i.e. $X = \bigcup_{i \in y} O_i$

Chapter 2: Simplicial complexes

Let $V = \{v_0, v_1, \dots, v_n\} \in E^n$ be a set of $n+1$ vertices/points in Euclidean n-space.

Def: $\text{aff}(V) := \{\lambda_0 v_0 + \lambda_1 x_1 + \dots \lambda_k x_k \mid \sum_{i=0}^{k} \lambda_i = 1,\ x_i \in R\}$
affine hull of V.

Ex



Remark: $\text{aff}(V)$ is an affine subspace of $\mathbb{E}^n$ (i.e. the solution set of a system of not necessarily homogeneous linear equations

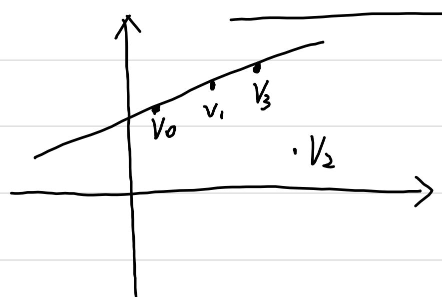
e.g. $x_1 + 2x_2 = 2$

Def: $\text{dimension}(\text{aff}(V)) := \dim(\text{span}(\{v_1 - v_0, v_2 - v_0, \dots v_k - v_0\}))$

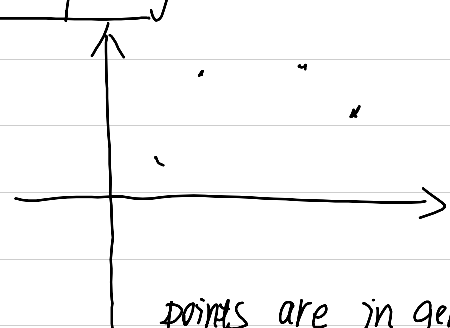
dimension of the affine hull

Def: $V = \{v_0, v_1, \dots v_k\}$ is in general position if no r of these points in $\mathbb{E}^n$ lie on a $(r-2)$-dim affine space for $r = 2, 3, \dots, n+1$

Ex:



not in general position

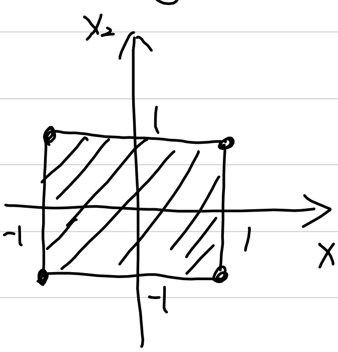


points are in general position.

: the line is the affine space

Def: $\text{convex}(V) := \{ \lambda_0 x_0 + \lambda_1 x_1 + \dots + \lambda_k x_k \mid$ convex hull of V $\sum_{i=1}^{k} \lambda_i = 1, \lambda_i \geq 0 \}$

Remark: convex(V) is the compact solution set of a system of not necessarily homogeneous linear inequalities. → the set of all variables that makes the equation true.



$x_1 \leq 1$

$-x_1 \leq 1$

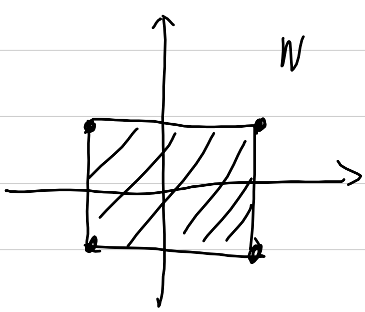
$x_2 \leq 1$

$-x_2 \leq 1$

Remark: convex(V) is called a (convex) polytope.

Def: Let $W \subseteq V$ be a subset of V that is affinely dependent and is of max. conventionality with this propery.
then $\dim \text{conv}(V) := \dim \text{aff}(W)$ is the dimension of the polytope conv(V)

Ex:



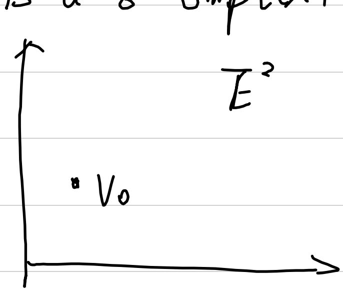
dim = 2

Def: If $V = \{v_0, v_1, \dots, v_k\} \subseteq E^n$ is affinely independent and $\delta \subseteq n$, then

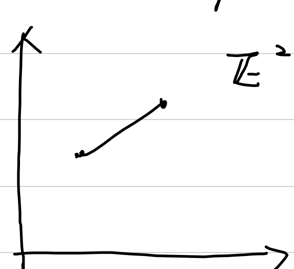
$$\delta := \text{conv}(V)$$

is a δ-simplex / δ-dimensional simplex

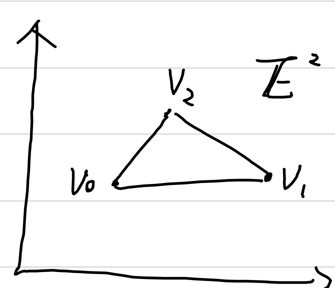
Ex



0-simplex



1-simplex



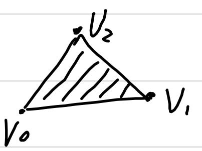
2-simplex.

Def: Let $\sigma = \text{conv}(V)$ be a simplex, then for any subset $W \subseteq U$

$$\tau = \text{conv}(W)$$

is a simplex too, called a face of σ, we write $\tau < \sigma$

Ex: $\sigma = \text{conv}(v_0, v_1, v_2)$



with faces

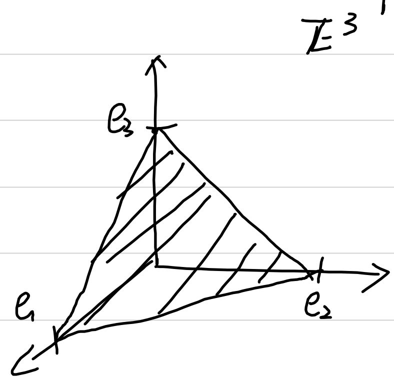
$\text{conv}(v_0, v_1, v_2)$

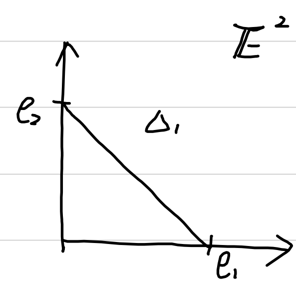
$\text{conv}(v_0, v_1)$, $\text{conv}(v_0, v_2)$, $\text{conv}(v_1, v_2)$, $\text{conv}(v_0)$, $\text{conv}(v_1)$, $\text{conv}(v_2)$

$\text{conv}(\emptyset)$ edges vertices

Def: A face τ of σ is proper if $\dim \tau < \dim \sigma$

Def: Standard $(n-1)$-simplex Δ_{n-1} in $\mathbb{E}^n$ is $\Delta_{n-1} = \text{conv}(e_1, e_2, \dots, e_n)$





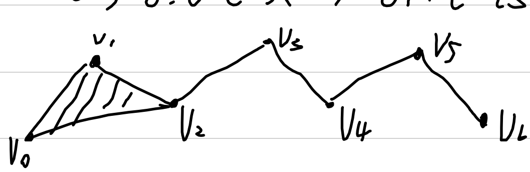
Geometric simplicial complexes.

Def: $\mathcal{K}$ (finite) geometric simplicial complex := (finite) collection of (geometric) simplices in some $\mathbb{E}^n$,

s.t. (1) $\sigma \in \mathcal{K}$ and τ a face of $\sigma \Rightarrow \tau \in \mathcal{K}$

(2) $\sigma, \tau \in \mathcal{K} \Rightarrow \sigma \cap \tau$ is a face of both σ and τ.

Ex:



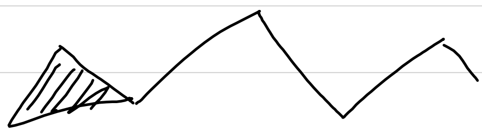
Ex

Def Let $\mathcal{K}$ be a (geometric) simplicial complex. that

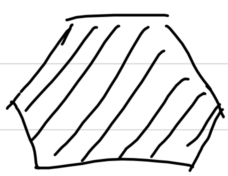
$$|\mathcal{K}| := \bigcup_{\sigma \in \mathcal{K}} \sigma$$

is the polyhedron of $\mathcal{K}$

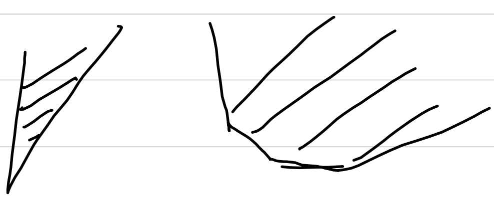
Ex:



Remark: Polyhedra in polytop theory are finite intersections of half-spaces and are convex:



polytope (bounded)



unbounded polyhedra

Aim: Define a topology on the set $|K| \subseteq \mathbb{E}^n$

Def: Let B be a collection of open sets in a topological space $(X\ \theta)$
If each open set in X is the union of open sets in $\mathcal{B}$, then the collection B is called a base of the topology O on X.

Ex: The collection of open balls $\dot{B}_r(x)$ in $\mathbb{E}^n$ forms a base for the topology of the Euclidean space $\mathbb{E}^n$

Def: (subspace topology) Let $Y \subseteq X$ be a subset of a topological space (X, θ)
Then $\theta_Y := \{ O \cap Y \mid O \in \theta_X \}$

defines a topology on Y, the subspace topology on Y (induced by the topology θ_X and X)

Week 2 Lecture 2

Def: For every polyhedraum $|K| \subseteq \mathbb{E}^n$
We equip $|K|$ (as a set) with the subspace topology interiored from $\mathbb{E}^n$

Remark: For a finite geometric simplicial complex, the inherited topology is "unique"
i.e. independent of the dimension of the Euclidean space.

Remark: The topology or $|K|$ is included by the base $\mathcal{B}$ of open balls in $\mathbb{E}^n$

Abstract Simplicial Complexes

Q: How can we store a geometric simplicial complex on a computer.

A: △ Store coordinates of the vertices $v_1, v_2, \dots, v_n$.

△ Store the list (collection) of faces. (i.e. convex hulls of vertices)

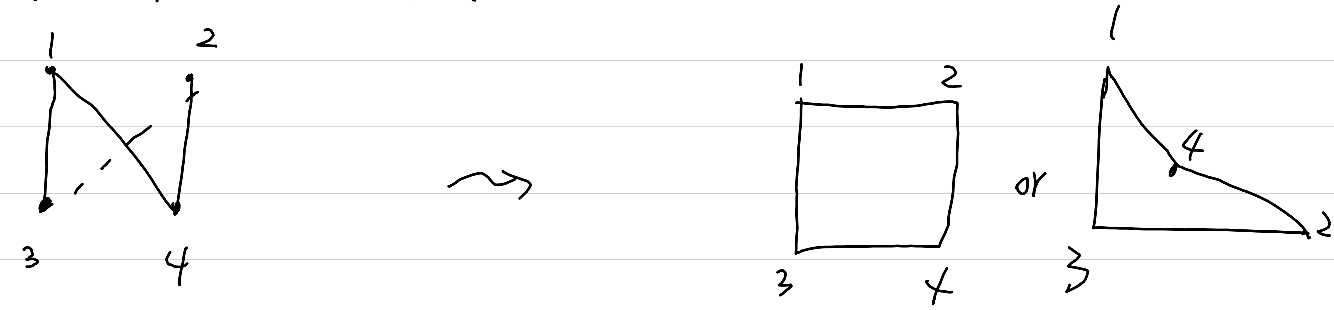
Def: (Abstract simplicial complex)

K is a (finite) abstract simplicial complex on a (finite) set $V = \{v_1, \dots, v_k\}$
if (1) $K \subseteq 2^V$
(2) If $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$

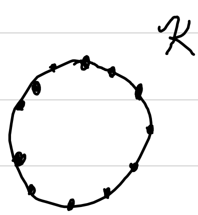
Remark: △ We write $V(K)$ to denote the set of vertices of K.
△ We often choose $V = \{1, \dots, k\}$ as the vertices.

Ex: $V = \{1, 2, 3, 4\}$
$K = \{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{2,4\}, \{1,3\}, \{1,4\}, \{2,3\}\}$

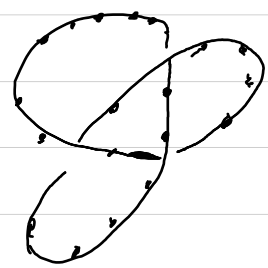
visualization/realization (see later) in $\mathbb{R}^n$



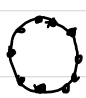
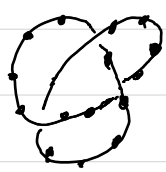
bad choice of vertices $\qquad$ without intersections

Ex: 

$S^1 \cong |K|$



trefoil knot in $\mathbb{E}^3$ (Kleeblattschlinge)

But $\mathbb{E}^3 \setminus$  $\not\cong \mathbb{E}^3 \setminus$  $\cong$ O

the unknot

The subdivided unknot (the standard circle) and the subdivided trefoil knot are
- isomorphic as simplicial complexes.
- homeomorphic as simplicial complexes

Def: (simplicial map)

Let K, L be (abstract) simplicial complexes with vertex set $V(K)$, $V(L)$

A map of $\varphi : V(K) \to V(L)$ is a simplicial map.

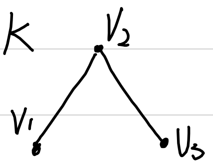
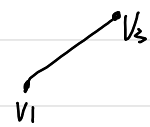
if $\forall S \in K : \varphi(S) \in L$

where $\varphi(S) := \{ \varphi(V_{i_1}) \ldots\ldots \varphi(V_{i_j}) \}$

for any $S = \{ V_{i_1} \ldots\ldots V_{i_j} \}$

Remark: $\triangle$ Simplices are mapped to simplices.

$\triangle$ The map $\tilde{\varphi}$ on the faces of K is induced by the vertex map φ.

Ex. K  L 

The map $\varphi: V(K) \to V(L)$

$$v_1 \mapsto v_1$$
$$v_2 \mapsto v_2$$
$$v_3 \mapsto v_3$$

The map $\varphi: V(L) \to V(K)$

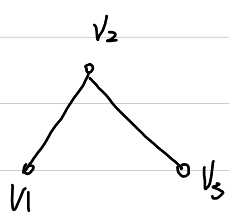
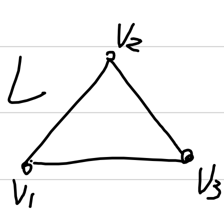
$$v_1 \to v_1$$
$$v_3 \to v_3$$

is not simplicial (as the edge $\{v_1, v_3\}$ in L is not mapped to an edge in K).

Def: Two simplicial complexes K and L are (combinatorially) isomorphic if there is a bijective map

$$\varphi: V(K) \to (L)$$

s.t. both φ and φ^{-1} are simplicial.

Ex: K  L 

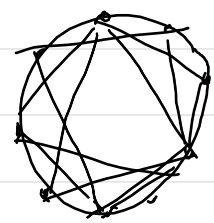
The (identity) map $\varphi: V(K) \to V(L)$

$$v_i \mapsto v_i$$

is bijective, simplicial, but φ^{-1} is not simplicial.

Remark: Graphs are 1-dimension simplicial complexes and graph isomorphism is hard to test (theoretically) but works pretty fast in practise.

Hard instances: e.g. regular graphs



Geometric Realization

Def: (Geometric Realization) We see that abstract simplicial complex K has a geometric simplicial complex $\mathcal{K}$ as a geometric realization if K and the underlying abstract simplicial complex of $\mathcal{K}$ are combinatorially isomorphic

Q: Does every abstract simplicial complex have a realization as a geometric simplicial complex?

A: Yes!

Let K be an abstract simplicial complex with $V(K) = \{v_1, \dots v_k\}$.

Define $\mathcal{K}$ to be the geometric simplicial complex in $\mathbb{Z}^n$

Triangulations of spaces

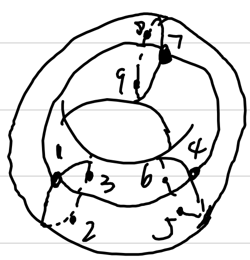
Def: The polyhedra of an abstract simplicial complex is

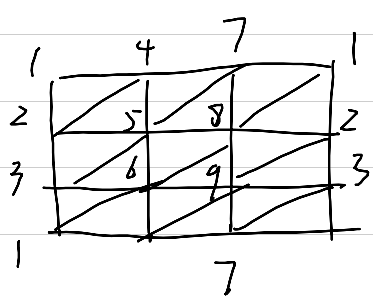
$$|K| := |\mathcal{K}| = |\varphi(K)|$$

for $\varphi : K \to \mathcal{K} \subseteq \Delta$

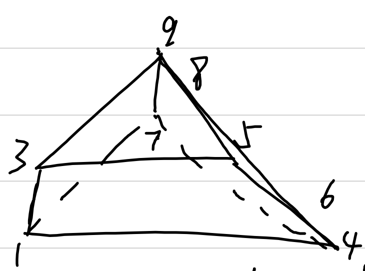
Def: Let X be a topological space and K be an abstract simplicial complex. We say that K is a triangulation of X resp. K triangulates X if $|K| = X$

Ex:





$\rightsquigarrow$ geometric realization in $\mathbb{E}^3$



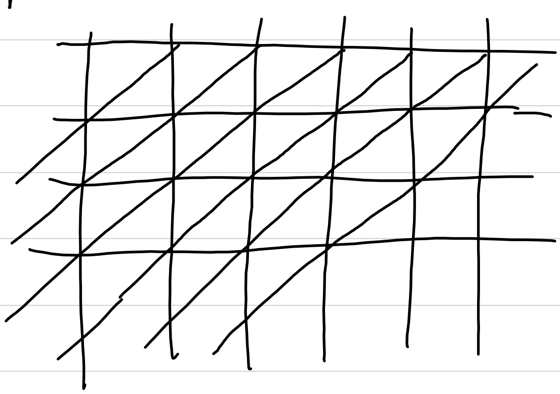
+ diagonals

Week 3 lecture 1

Triangulation: $\varphi: K \to X$ while $|K| \cong X$

Remark: Not every topological space can be triangulated, as a (finite) abstract simplicial complex, e.g. there are compact d-manifolds for any $d \geq 4$ that cannot be triangulated.

We can generalize finite abstract simplicial complexes in order to triangulate non-compact spaces.



An infinite simplicial complex K.

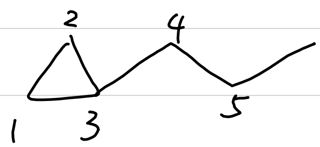
- Δ is of finite dimension,
  if the dimension of the simplices of complex is bounded.
- Δ is locally limited
  is contained in at most finitely many simplices.

Remark: All simplical complexes we consider are locally finite and of finite dimension.

Facet description of simplicial complexes

Def: A face σ of a geometry/abstract simplicial complex is a facet if σ is the unique face of the complex it is contained in.

Remarks: Facets are inclusion maximal faces.

Ex:  has facets $\{123, 34, 45, 56\}$

Remarks: Facets can be of different cardinality/dimension.

Def: A simplicial complex K is pure if the facets of K are all of the same dimension.

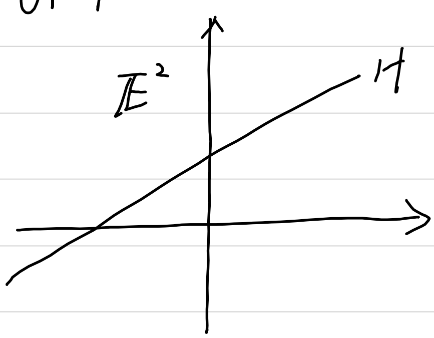
Remark: The list of facets of K contains all the information on K. Thus, the facet description of K is a compact format to store K on a computer.

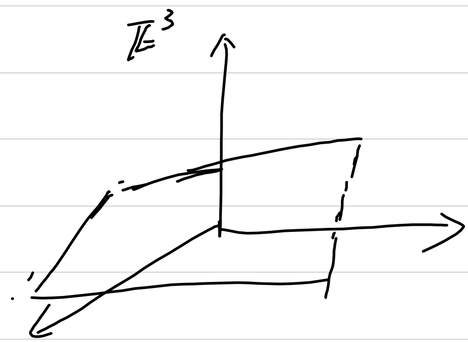
Chapter 3 Realizability

Def: Let $X \subseteq \mathbb{E}^n$ be a subset of Euclidean n-space

- A point $x \in X$ is an interior point of X if there is an open ball $\mathring{B}_r(x)$ centered at x that is fully contained in X, i.e. $x \in \mathring{B}_r(x) \subseteq X$
- The interior $\mathring{X}$ is the union of all interior points of X.
- A point $x \in X$ [or $\bar{X}$] is a boundary point of X, if $x \notin X^\circ$ ($\bar{X}$ closure)
- The boundary ∂X is the union of all boundary points of X.

Def: A hyperplane H is an $(n-1)$ dimension affine subspace of $\mathbb{E}^n$





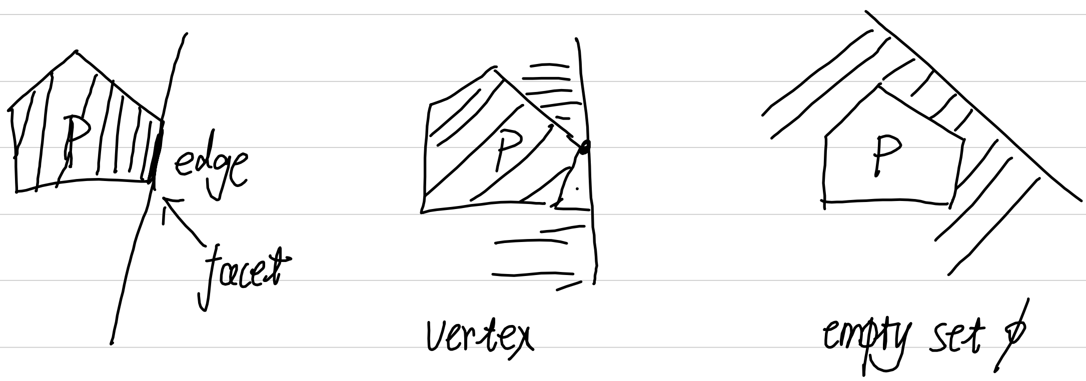
Def: Let $X \subseteq \mathbb{E}^n$

A hyperplane H is supporting for X if H bounds a half-space of $\mathbb{E}^n$ that contains X.

Def: Let $P \in \mathbb{E}^n$ be an n-polytope.

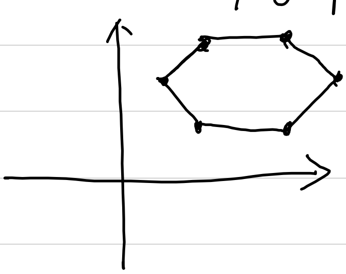
- △ A face of P is the intersection $P \cap H$ of P with a supporting hyperplane H.

- △ A facet of P is an $(n-1)$ dimmension face.

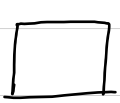


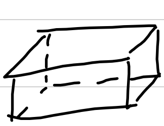
- △ ∂P consists of the collection/union of faces of P.

Def: △ A polytope P is simplicial if all of its faces are simplices
 ○ A polytope P is cubical if all of its faces are cubes (of varying dimension)

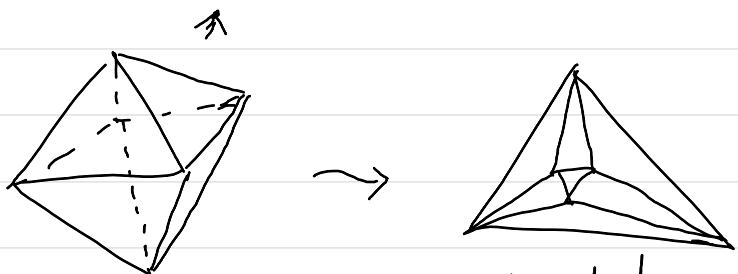


 1-cube

 2-cube

 3-cube

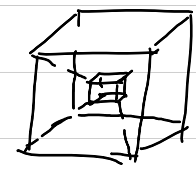
Def: A Schlegel diagram of a polytope P is the projection of P onto one of its facets trough a point that just lies outside of facet.



the octatedron is a simplicial 3-polytope

Schlegel diagram

Ex: Schlegel diagrams of a k-cube



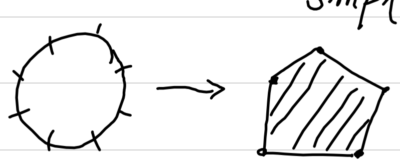
Remark: Let P be simplicial. then

- ∂P is a simplicial complex that triangulates
- any Schlegel diagram of P along with the projection facet gives basic ∂P (as an abstract simplicial complex)

Thm: (Stuniz theorem) [Steinitz, 1916]
Every abstract simplicial complex that triangulates S^2 can be realized as the boundary complex ∂P of a simplicial 3-polytope.

Def: A triangulation K of S^n is polytope complex ∂P of a, simplex $(n+1)$ polytope.

Ex: K triangulates S^1



Thm: [Bokowski Gamms 1987] All schlegel's 3-sphere M^{10}_{425} with

10 vertices is not polytope.

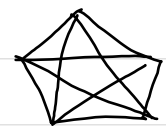
Corallary: For $n \geq 3$, there are non-polytopal triangulations of S^n.

Thm [Mnev 1988, Richter-Gebert 1995, Bokowski & Sturmfels 1989]

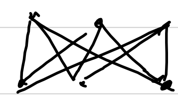
It is NP-hard to decide whether a given triangulation of S^n, $n \geq 3$ is polytopal.

Every 1-dimensional simplicial complex (graph)

can be realized geometrically in $\mathbb{R}^3$, but graphs containing K_5



or K_3



as a minor are not planer i.e. <u>cannot be realized geometrically</u> in $\mathbb{R}^2$.

(not even embedded)

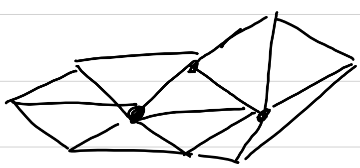
Tasting plannarity lies in $O(|V|)$ i.e. has linear running time

Week 4 Lecture 1

→ Combinatorial properties of triangulated manifolds.

Requirement: pure

Ex: Some part of a triangulation K of a surface.



Every point $x \in |K|$ has a neighbourhood that looks like an open disc $\mathring{B}^2$

x can lie <1> in the interior of a triangle

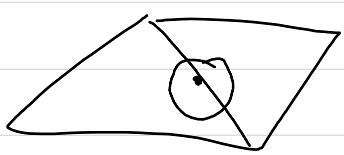
<2> on an edge

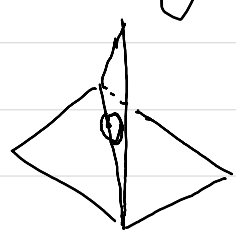
or is <3> a vertex

$\overline{\text{star}}(x) = \mathring{\text{star}}(x) \cup \text{link}(x)$

<1> Interior points of triangles have small discs around them

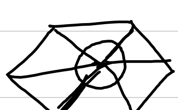
<2> For a point on an edge every incident triangle contributes half-disc





not locally homeomorphic to $\mathbb{R}^2$

<3> For a vertex, the incident triangles have to form a disc.



link(x)

$\mathring{\text{star}}(x)$ open star.

Def (Open Star):

Let σ be a face of K

Then $\mathring{star}(\sigma) = \{\tau \in K : \sigma \subseteq \tau\}$

is the open star of σ in K.

Remark: $\mathring{star}(x)$ is in general not a simplicial complex.

Def: (closed star)

Let σ be a face of K

Then $\overline{star(\sigma)} = \{\tau \in K \mid \tau \cup \sigma \in K\}$

is the closed star of σ in K.

Def (link)

Let σ be a face of K. then

$link(\sigma) = \{\tau \in K : \tau \cup \sigma \in K, \tau \cap \sigma = \emptyset\}$

Remark: $\overline{star}(\sigma)$ and $link(\sigma)$ are subcomplexes of K.

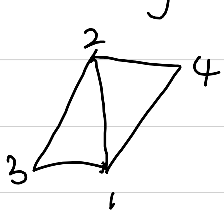
Def (subcomplexes)

Let K be a simplicial complex and $L \subseteq K$ s.t. L is a simplicial complex.

then L is a subcomplex of K.

Notation: We often use $star(\sigma)$ to denote the closed star of σ.

Ex:



$link(\{1,2\}) = \{\emptyset, \{3\}, \{4\}\}$

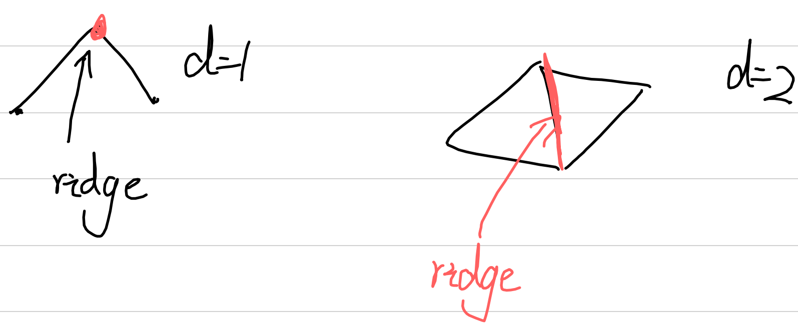
or stout-hand in facet description

$link(12) = \{3, 4\}$

$star(12) = \{123, 124\}$

Def: Let K be a pure simplicial-complex. A $(d-1)$-dimensional face is called a ridge.

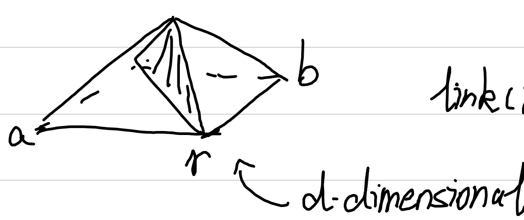
Ex



Lemma: Let K be triangulation of a (compact) d-manifold (without boundary) as a (finite) simplicial complex and let $r \in K$ be a ridge. then

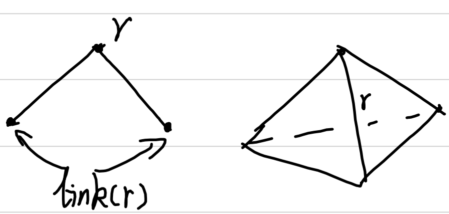
$$\text{link}(r) = S^0$$

Proof:



$\text{link}(r) = \{a, b\}$ "$\cong$" S^0

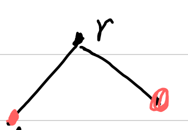
Ex:



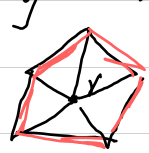
Remark: In a triangulation of a manifold (without boundary), a ridge lies in exactly two facets.

Q: What do we know for links of vertices?

$d=1$:



$\text{link}(r) = S^0$



$\text{link}(r) \cong S^1$

In higher dimensions:

For $d = 3, 4$: $\mathrm{link}(r) \cong S^{d-1}$

For $d \geq 5$: the situation is more complicated.

Def: A topological space X is path-connected, if for any $x, y \in X$ there is a path that connects x and y.

Theorem: A connected manifold is path-connected.

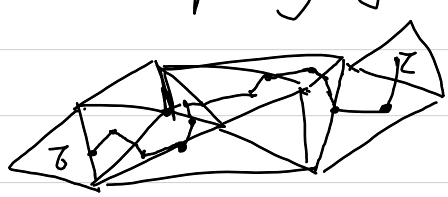
Def: A pure d-dimensional simplicial complex is strongly connected if for every point of facets σ and τ there is a sequence

$$\sigma = \sigma_1 \, r_{12} \, \sigma \, r_{23} \, \sigma_3 \, \dots \, r_{j-1\,j} \, \underset{\tau}{\underset{\shortparallel}{\sigma_j}}$$

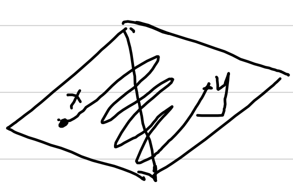
of facets, $\sigma_1, \dots, \sigma_j$ and ridges $r_{12}, \dots, r_{j-1\,j}$

s.t. $r_{i-1\,i}$ lies in σ_{i-1} and σ_i

i.e. there is a path of facets from σ to τ that goes via ridges.



Lemma: Path-connected triangulations of closed manifolds are strong connected.



"Proof" (idea). A path is a continuous map: $P: I \to X$

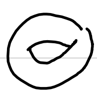
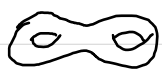
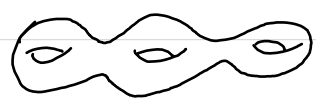
(unit) interval.

The continous image of a compact space is compact.

A polygonal path P in X can revisit a facet only finitely many times.
We can shorten respectively omit then revisits to obtain a path that goes via each facet only once and accomodate it that it goes <u>via ridges</u>

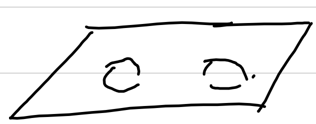
Chapter 5 Classification of (closed) surfaces

In our zoo of surfaces (2-dim. manifolds)
We have  S^2

 T^2  

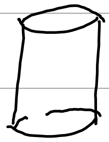
$\mathbb{RP}^2$ Klein bottle

$\mathbb{E}^2$ $\mathbb{E}^2 \setminus \overline{B}^2$

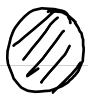
 

$\cong \mathbb{E}^2 \setminus pt$ punctured place

 $\cong \mathbb{E}^2 \setminus pt$ open cylinder



closed cylinder



closed disc

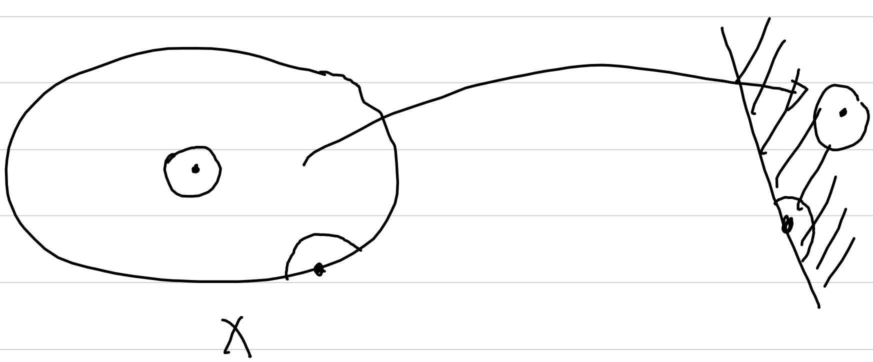
open disc $\cong \mathbb{E}^2$

As a model-space for manifolds-with-boundary

Def: A (Hausdorf and second-countable) space M^d is a d-manifold with or without boundary if for every $x \in M^d$ there is a open neighbourhood U_x of x s.t. U_x is homeomorphic

- to an open d-ball in the closed half-space $\mathbb{E}^d_+$
- to an open half-ball

In the first case. x is an interior point.
in the second case, x is a boundary point.



Week 4 Lecture 2

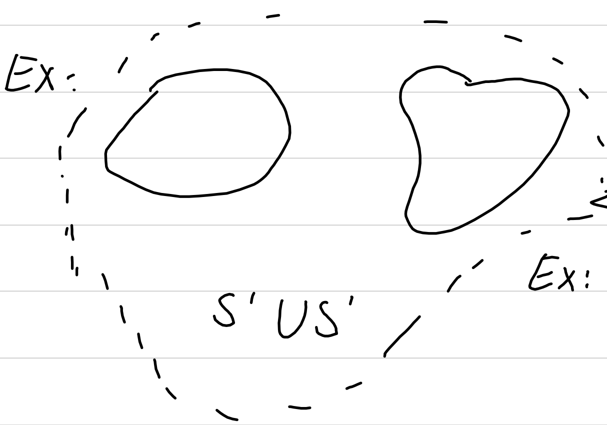
Def: A d-dimensional manifold without boundary is
- closed if it is compact
- open if it has non-compact components.

From now on we assumed a manifold to be closed and connected.
(unless we consider special cases such as
- $S^0 = \circ$
- ∂(closed cylinder) $=$ O S^1
  O S^1

Thm: [Rado 1925]
closed 2-manifolds can be triangulated (as finite simplicial manifolds)

Thm: Every triangulation of a connected closed manifold is strongly connected.

Ex:



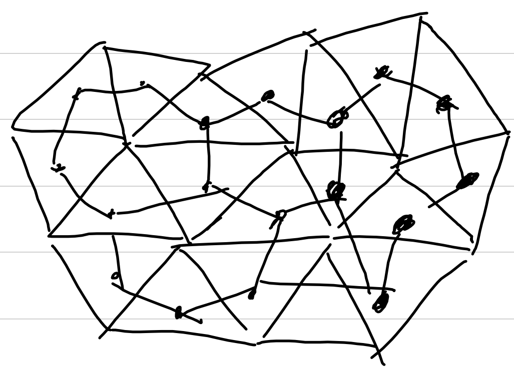
closed manifold that is not strongly connected

Ex: A simplicial complex that is strongly connected need not be a triangulation of a manifold.

link(o) = O O

pinch points $\to$ pseudo-manifolds.

Def: The dual graph of a triangulated (connected and closed) d-manifold M^d consists of the facets of M^d as vertices of the graph and edges if two facets have a common ridge.



Q: Does the dual graph determine the triangulation?

A: By the degree of a vertex in the dual graph we know the dimension. so we load some information, but distinct triangulations could have the same dual graph.

Q: How to determine whether two triangulated d-manifolds are isomorphism.

- △ Check number of vertices
- △ check degree of vertices. △ check number of facets
- △ Every facet of the "first" triangulation K has to be mapped to a facet of the "second" triangulation L (if K and L are isomorphic)

pick some triangulations.

Number of possible choices: # triangles $(d+1)!$

determine all triangulation, by strong connectivity.

K

L

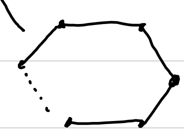
fixed.

- neighbouring triangle are mapped to neighbouring triangle.

Remark: The implementation of the isomorphic check allows to also compute the combinatorial isomorphism group Aut(K) of a triangulated manifold K.

Ex:

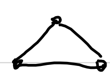
- $\mathrm{Aut}(C_n) = D_n$ with $|D_n| = 2n$

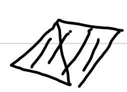
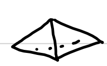


n vertices

- $\mathrm{Aut}(\partial\Delta_n) = S_{n+1}$ Symmetric group on n+1 elements.

$\Delta_1 =$  $\partial\Delta_1 =$

$\Delta_2 =$  $\partial\Delta_2 =$ 

$\Delta_3 =$  $\partial\Delta_3 =$ 

Classification procedure (algorithm)

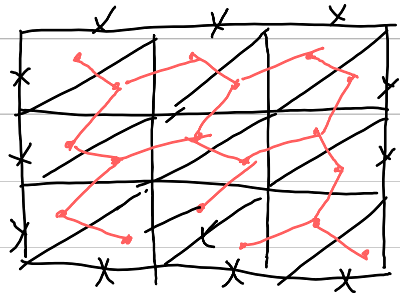
Input: triangulate of a closed, connected surface

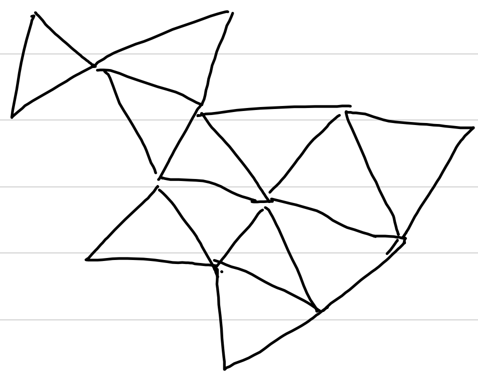
Output: type of the surface.

1. Step Pick a spanning tree in the dual graph of the triangulation.

Remark: The dual graph is connected, because the triangulation is assumed to be strongly-connected.

2. Step Cut open all edges of the triangulation, that are not crossed by an edge of the spanning tree → in the dual graph.

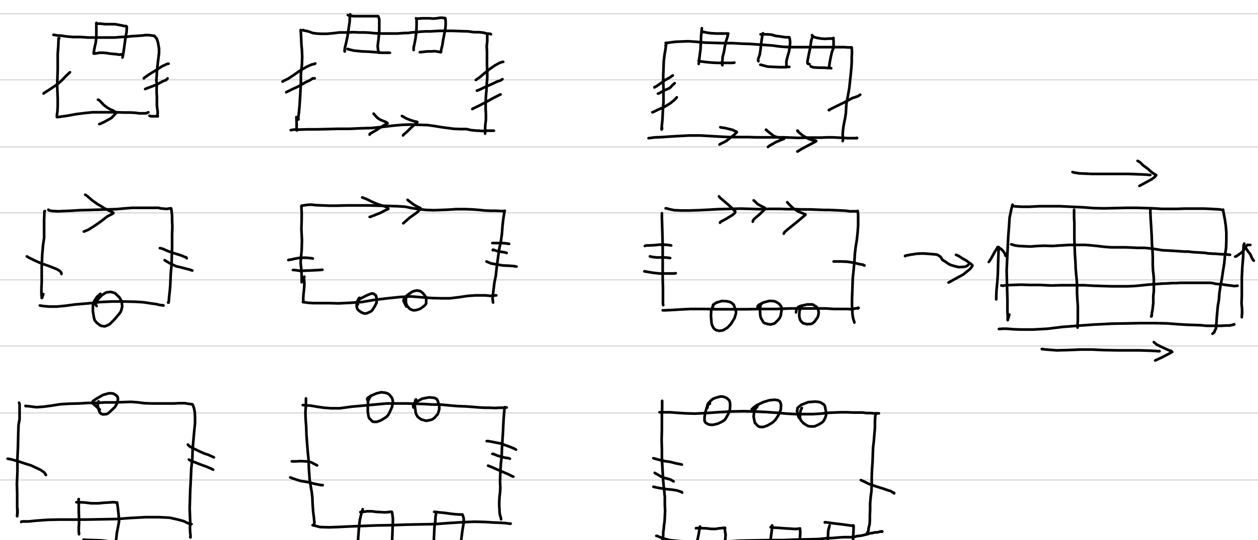




$\rightsquigarrow$ triangulation of a disc with the extra property than boundary edges are pairwise identified.

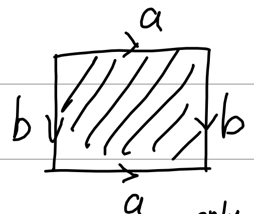
Def: Polygonal decomposition / scheme of a (closed, connected) surface is a decomposition into a (finite) family of polygons with pairwise identified edges.

Ex:



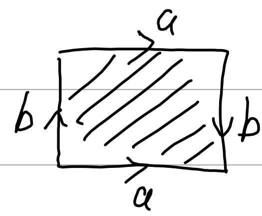
Remark: Polygons of a decomposition can be glued together along edges to form a simple polygon with positive identified edges.

Ex

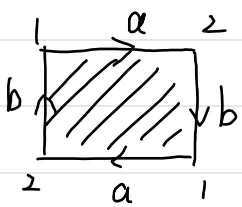


only one vertice

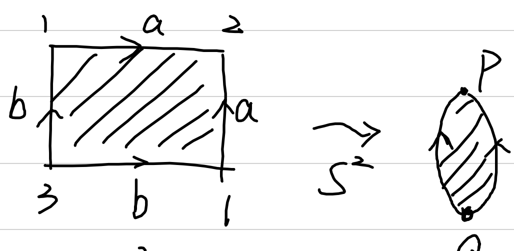
T^2



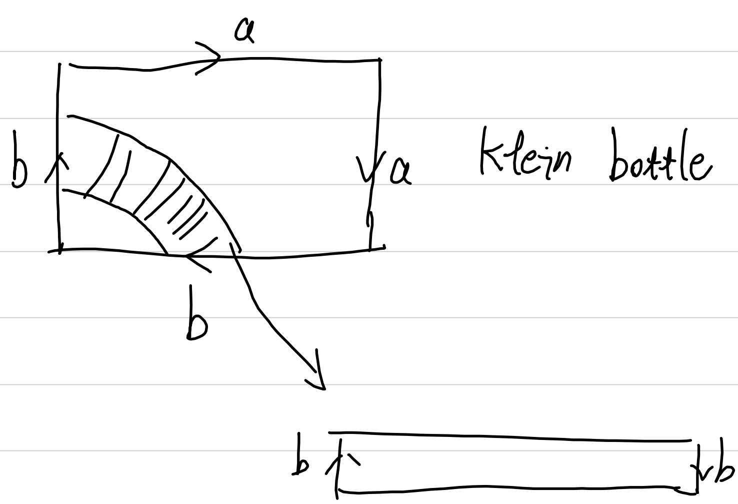
Klein bottle



$\mathbb{R}P^2$



S^2



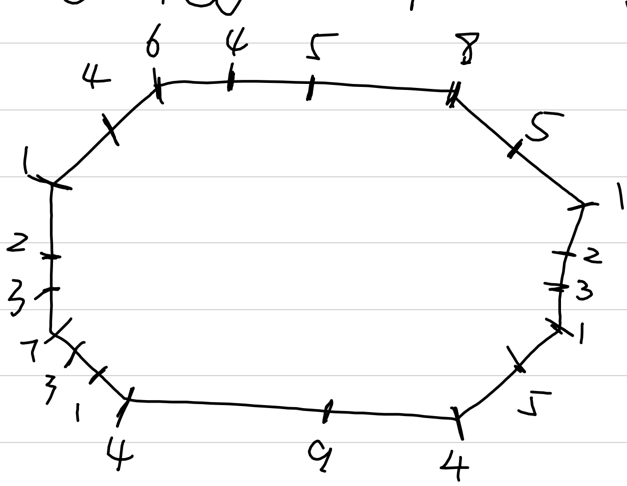
Klein bottle

Möbius banned / strip

$\rightsquigarrow$ non-orientiable

3 Step. For the previous tree of triangles remove the interval edges to obtain a single polygon with positive identified edges.

Ex:



4. Step Reduce such a sheme to a scheme with only one (or two) vertices

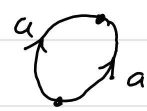
$\rightsquigarrow$ simplification of schemes!

Week 5 Lecture 1

Aim: Simplification of schemes.

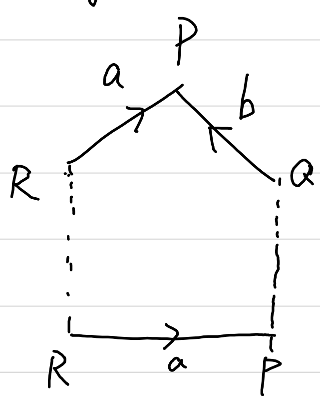
Prop: Every scheme of a surface can be reduced to

- the sheme aa^{-1} of S^2

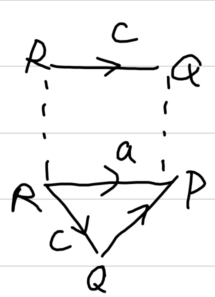


- or to a sheme with exactly one vertex.

Proof: Let the scheme have at least two different equivalent classes of identified vertices.



P and Q are different vertices with elements in the equivalent class of P.

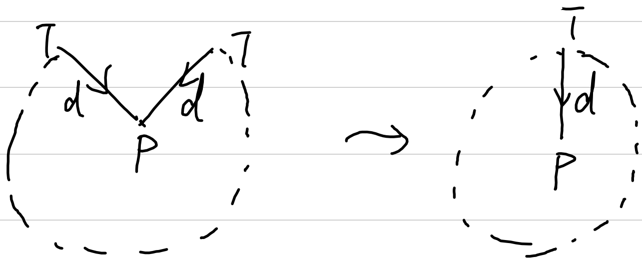


m-1 elements in the equivalent class of P (through the size of the class of Q has increased)

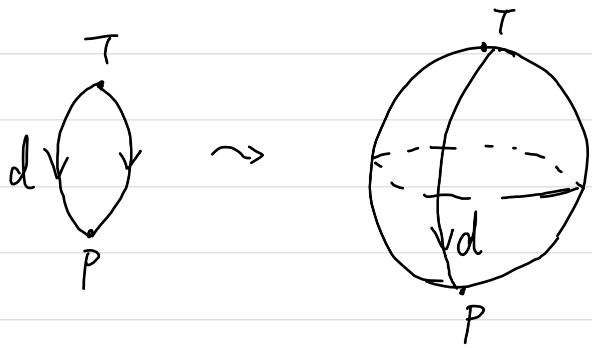
$\rightsquigarrow$ eventually

P appears only once on the baundary of the (modified) scheme.

two cases:

1. 

remove internal P and d.

2. 

For 1. After the detection of further equivalent classes we obtain either.

- scheme dd^{-1} of S^2
- or scheme with exactly one vertex.

5. step Transformation into canonical form.

From now on: scheme has exactly one vertex. We simplify the scheme further by

Ⓐ cross-up normalization

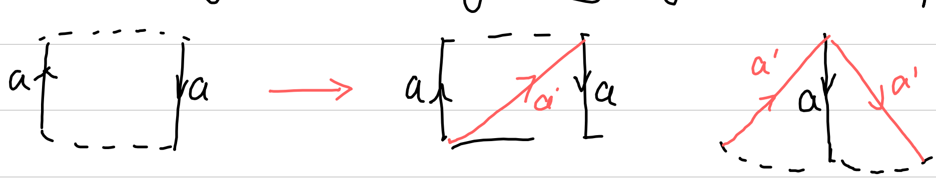
Ⓑ handle normalization

Ⓒ transformation of handles into cross-caps (in case both one present)

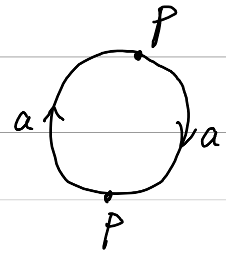
Ⓐ cross-up normalization.

$\ldots a \ldots a \ldots \rightsquigarrow \ldots a' a' \ldots$

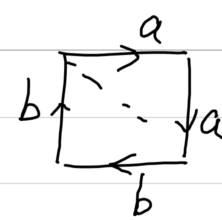
$\rightsquigarrow$ Pairs of identified edges of the same orientation on the boundary of a scheme can be transformed into neighbouring edges (cross caps)

Proof: 

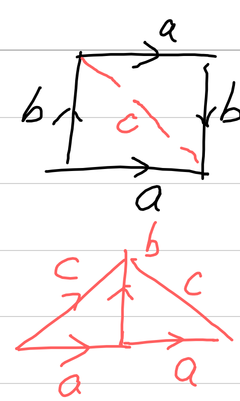
$\mathbb{R}P^2$



Klein bottle



Klein bottle.



(B) Handle normalization

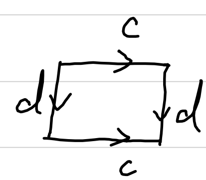
Prop: After cross-up normalization

(i) pairs of oppositely oriented edges occur as crossed pairs.

$\cdots a \ldots b \ldots a^{-1} \ldots b^{-1}$

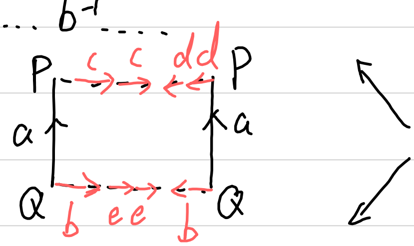
and can be transformed to $\ldots c d c^{-1} d^{-1} \ldots$

tour:



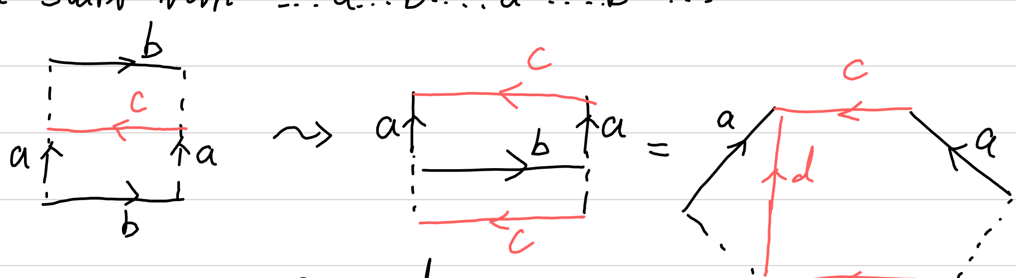
Proof: (i) Let us assume $\ldots a \ldots a^{-1} \ldots$ is not separated by a pair $\ldots b \ldots b^{-1} \ldots$

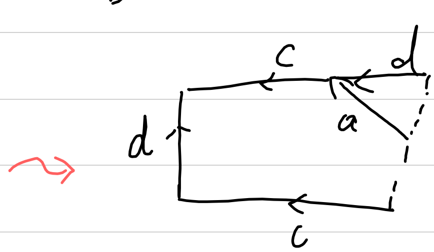
Then



P and Q are not identified

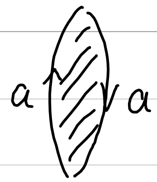
(ii) we start with $\ldots a \ldots b \ldots a^{-1} \ldots b^{-1} \ldots$





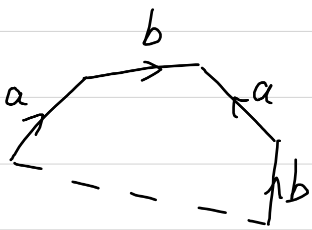
Special configurations.

$\mathbb{RP}^2$



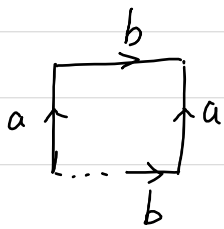
cross up

handle

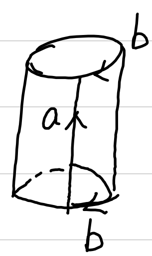


$\cdots a \cdots b \cdots a^{-1} \cdots b^{-1} \cdots$

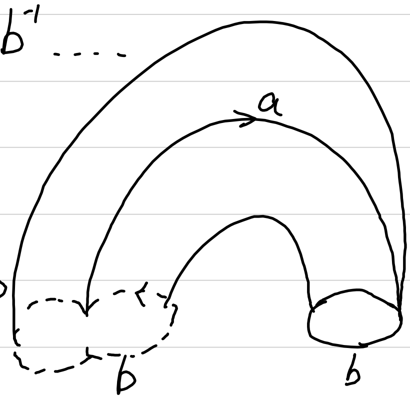
$\rightsquigarrow$



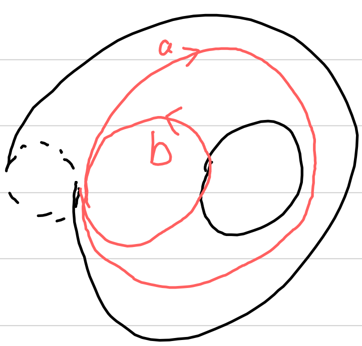
$\rightsquigarrow$



$\rightsquigarrow$

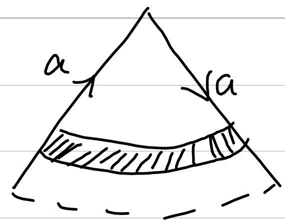


$\rightsquigarrow$



$\rightsquigarrow$

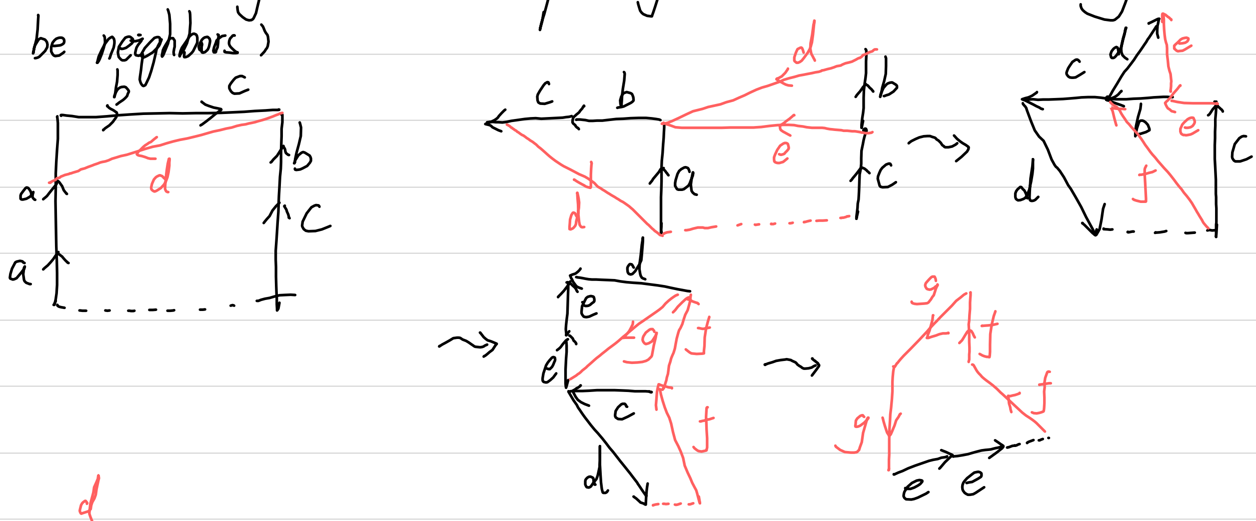
Remark: In case there is a cross cap



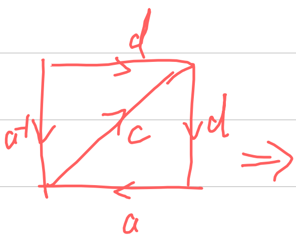
there is always a Mobius banned which forces the surface to be non-orientable (through the handles themselves are orientable).

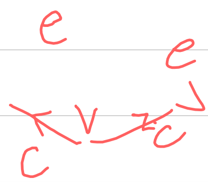
(c) Transformation of handles into cross caps in the presence of both handles and cross caps.

If there is at least one cross cap and at least one handle, then the boundary of the polygon of the scheme has a subsequence ... $aabcb^{-1}c^{-1}$.... (Otherwise we just have cross caps or just handles — somewhere they need to be neighbors)



$\dots e^{-1}e^{-1}g^{-1}g^{-1}f^{-1}f^{-1}\dots$



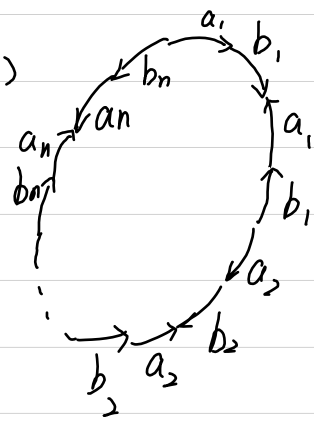


Thm [Brahama 1921]

Every (closed, connected) surface can be transformed into one of the following schemes:

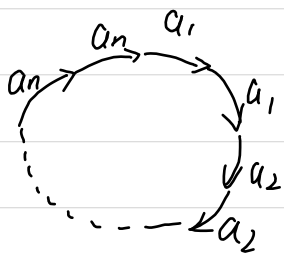
(i) 

sphere S^2

(ii) 

→ handles

} orientable

(iii) 

n cross cap } non-orientable

Remark: Such a scheme is called the normal form of a surface.

Q: Does the theorem of Brahama give us a classification of surfaces?

A: Not yet!

It could still be the case thus some of the normal forms represent the same surface. → We need to distinguish between the different normal forms.

Def: A topological invariant is a map that assigns to each topological space $X \in \text{Top}$ some cases.

$I : \text{Top} \to \{ \text{Groups}, \mathbb{R}, \vdots$

topology of topological spaces

If a $\begin{Bmatrix} \text{group} \\ \text{number} \end{Bmatrix}$ is assigned, then I is called an $\begin{Bmatrix} \text{algebraic} \\ \text{numerical} \end{Bmatrix}$ invariant

It is required that if $X \cong Y$, then $I(X) \cong I(Y)$

isomorphic.

i.e. homeomorphic spaces have the same invariant I.

Examples:

- dimension is numerical invariant for manifolds.
- genus is numerical invariant for surfaces.
- connectivity . . .

Week 5 lecture 2

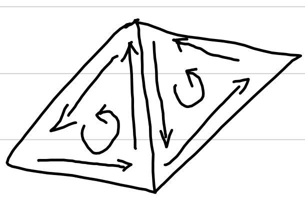
Def: A complete classification of a family of topological spaces $\{X_j, j \in J\}$ for some index set J. is a partition into homomorphic spaces.
e.g. by specifying some list of topological intervals. $I_1, I_2 \dots I_n$ that together allow to distinguish between non-homomorphic spaces in the family.

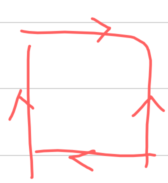
For surfaces
- Euler characteristics + orientability
- or homology

Orientability: in math, orientability is a property of some topological spaces such as real vector spaces, Euclidean spaces, surfaces, and more generally manifolds that allows a consistent definition of "clockwise" and "counterclockwise"

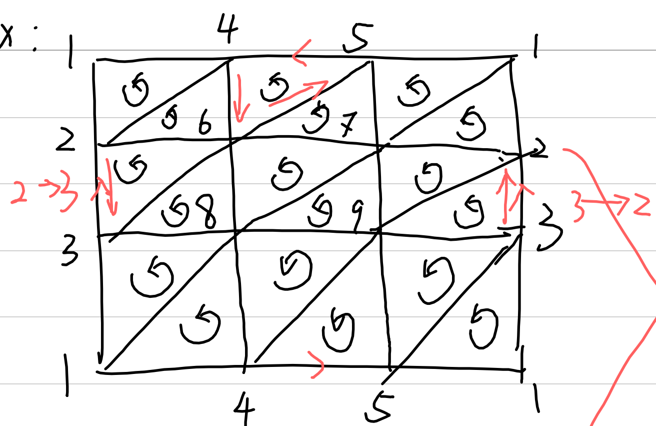
Def: A triangulated surface is orientable if the triangles of the triangulation can be oriented coherently s.t. each ridge inherits opposite orientations from its two neighbouring triangles.

If a coherent orient. does not exist, then the triangle is non-orientable.

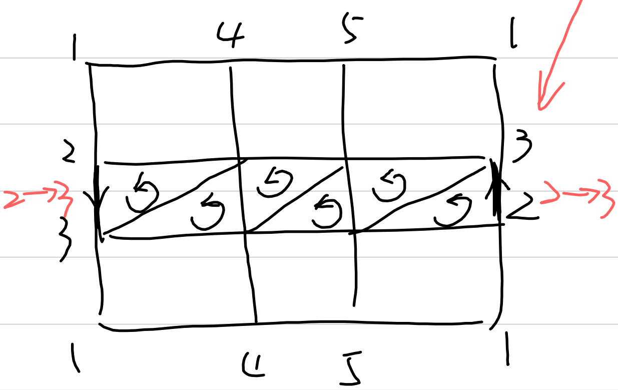




Ex:



Klein bottle is not orientable.



The Mobius strip that we found have is an destinction to orientability!

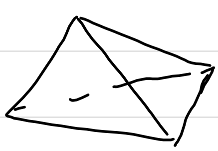
The Euler characteristic of a surface.

Def: For a triangulation (or more general for a polygonal decomposition) of a surface, Let

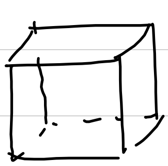
$$\chi = \#\text{ vertices} - \#\text{ edges} + \#\text{ faces}$$

be the Euler characteristic of the decomposition.

Ex:



$\chi = 4 - 6 + 4 = 2.$



$\chi = 8 - 12 + 6 = 2$

Then (Euler's polyhedron formula)

Let P be a 3-polytope with V vertices, E edges, F faces.
then $V-E+F=2$.

Q: Let a surface M^2 have triangulations K and K'.
Is then $\chi(K)=\chi(K')$?
I.e. is χ a topological invariants?

Thm: [Kerékjártó 1923]
Let K and K' be two triangulations of a (closed, connected) surface M^2.
Then K and K' have a common subdivision K'', obtained from K resp. K' by sequences of face and edge subdivisions.

$$\begin{tikzcd} K \arrow[dr] & & K' \arrow[dl] \\ & K'' & \end{tikzcd}$$

$\rightsquigarrow$ true also in dimension $d=3$
false in general for $d\geq 4$

$\rightsquigarrow$ There are topological d-manifolds for $d\geq 4$ for which there are non-equivalent PL structures.

$\rightsquigarrow$ Open: Does S^4 have exotic structures?

Face subdivisions:

$\chi - 1 + 1 - 3 + 3 = \chi.$

reserved triangle   new vertex   new edges   new triangles

Edge subdivison:

$\chi + 1 - 2 + 1 - 4 + 4 = \chi$

new 4 edges

Corallary: The orientability character and the Euler character are topological invariants for surfaces (and also for higher-dim manifolds.)

topology:

$H_{\mathbb{Z}}(M^2) = (H_0(M^2), H_1(M^2), H_2(M^2))$

homology $= (\mathbb{Z}^{\beta_0}, \mathbb{Z}^{\beta_1} \oplus T, \mathbb{Z}^{\beta_2})$

- $\chi = \beta_0 - \beta_1 + \beta_2$
- $\beta_0 =$ # connected components.
- β_1 related to the genus (in case $\beta_0 = 1$)
- $\beta_2 = \begin{cases} 0 & \text{if non-orientable} \\ 1 & \text{if orientable} \end{cases}$ (in case $\beta_0 = 1$)

△ $\chi(M^2) = (\gamma_0, \gamma_1, \gamma_2)$

Betti vector Betti numbers

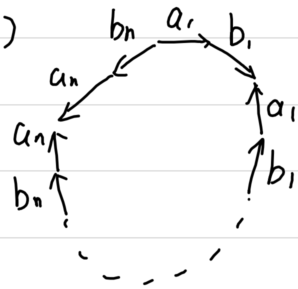
Thm (Classification of surfaces)

Every surface is of one of the following types:

(i)  S^2 $\chi = 2-1+1 = 2$

$H_{\mathbb{Z}} = (\mathbb{Z}, 0, \mathbb{Z})$

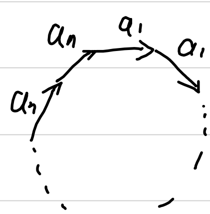
orientable

(ii) 

n handles $\chi = 1-2n+1 = 2-2n$

$H_{\mathbb{Z}} = (\mathbb{Z}, \mathbb{Z}^{2n}, \mathbb{Z})$

orientable

(iii) 

n cross caps

$\chi = 1-n+1 = 2-n$

$H_{\mathbb{Z}} = (\mathbb{Z}, \mathbb{Z}^{n-1} \oplus \mathbb{Z}_2, 0)$

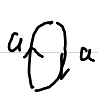
$[H_{\mathbb{Z}}(-;\mathbb{Z}_2) = (\mathbb{Z}_2, \mathbb{Z}_2^n, \mathbb{Z}_2)]$

$[H_{\mathbb{Z}}(-;\mathbb{Z}_3) = (\mathbb{Z}_3, \mathbb{Z}_3^{n-1}, 0)]$

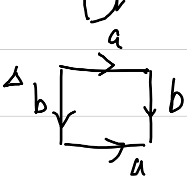
non-orientable

Remark: Euler characteristic + oriented characteristic give a complete classification of surfaces.

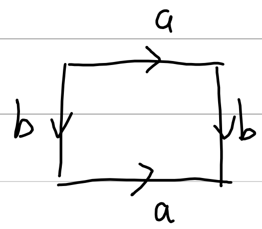
Alternatively: homology.

Examples: △ 

$\mathbb{R}P^2$ $\chi = 1-1+1=1$ $H_{\mathbb{Z}} = (\mathbb{Z}, \mathbb{Z}_2, 0)$

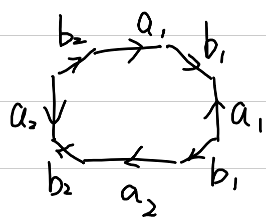
△ 

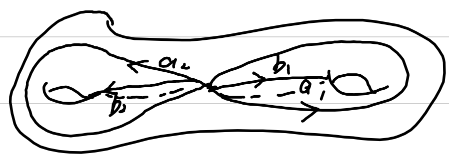
$\chi = 1-2+1 = 0$ $H_{\mathbb{Z}} = (\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z})$

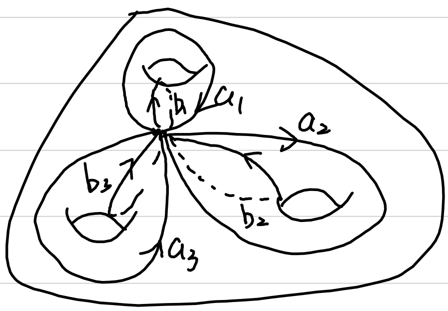
- 

Klein bottle $\chi = 1-2+1 = 0$

$H_* = (\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2, 0)$

- 



- 

Def: Orientable surfaces with $\chi = 2-2n$ the genus is

$$g = n$$

For non-orientable surfaces with $\chi = 2-n$ the (non-orientable) genus is

$$M = n.$$

In differentiable geometry:

- Spherical surfaces, $\chi > 0$ S^2, $\mathbb{R}P^2$ (as quotiants of S^2)
- flat surfaces, $\chi = 0$ T^2, Klein bottle (as quotiants of $\mathbb{R}^2$)
- Hyperbolic surfaces, $\chi < 0$ all others (as quotiants of hyperbolic space H^3.

For 2-manifolds: three model geometry which is (S^2, E^2, H^2)

For 3-manifolds, 8 model geometry $[S^3, E^3, H^3, \dots]$

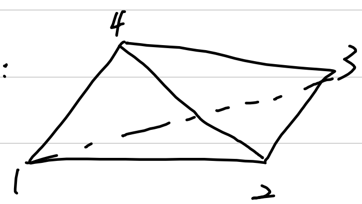
geometrisation of 3-manifold

spherical Euclidean hyperbolic

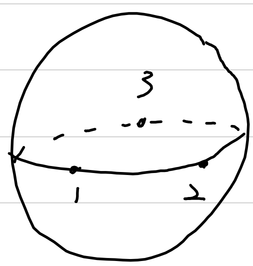
$\mathbb{R}^2$ with different metric

Chapter 6 Vertex-minimal triangulations of surfaces.

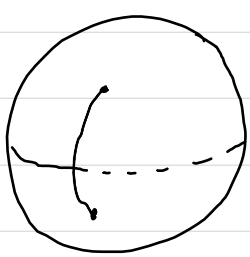
Q: How many vertices do we need to triangulate a surface M^2 (as a simplicial complex)?

Ex: 

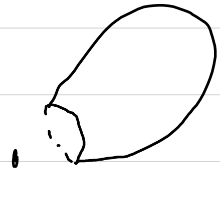
S^2 4 vertices.

[

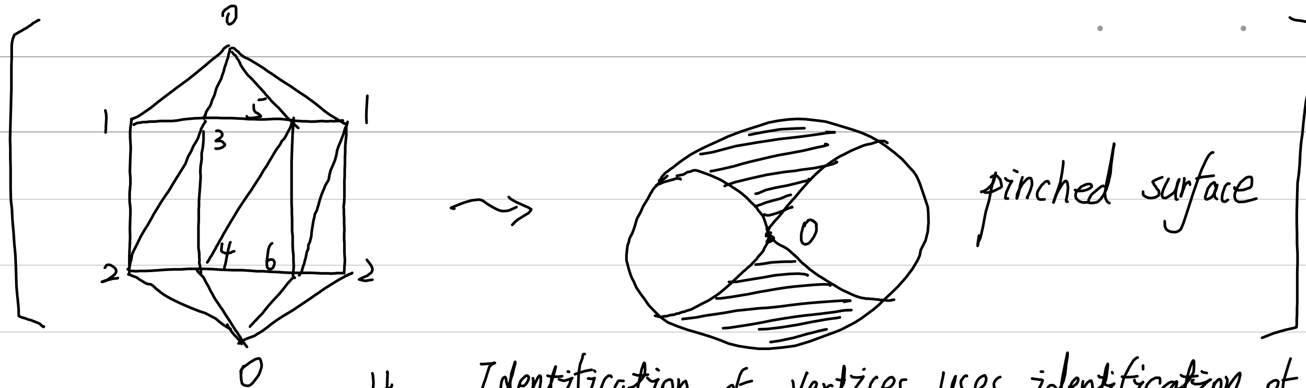
3 vertices
pseudo-simplicial complex (all faces are simplices)]

[

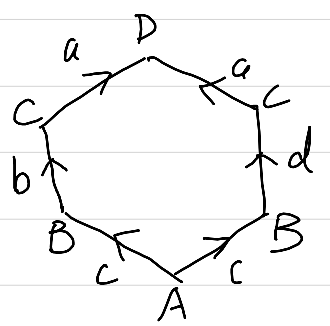
2 vertices scheme]

[

1 vertex CW complex]



Here: Identification of vertices uses identification of edges.
vs. Identifications of edges induce identify of vertices.



S^2

Definition: K simplicial d-complex $f_i :=$ # i-dimensional faces of K.

$f = (f_0, f_1, \ldots, f_d)$ f-vector or face vector of K.

$$\chi(K) := \sum_{i=0}^{d} (-1)^i f_i \quad \text{Euler characteristic.}$$

From topology: $\triangle$ χ is a topological invariant.

$\triangle$ $\chi(M^d) = 0$ for any odd-dimensional manifold M^d.

f-vector of a surface:

$$f = (V, E, F) = (f_0, f_1, f_2) = (n, f_1, f_2) = (n, f_1, \tfrac{2}{3} f_1)$$

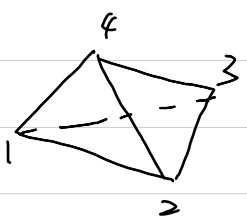
vertices   see below

Def: (Hasse diagram)

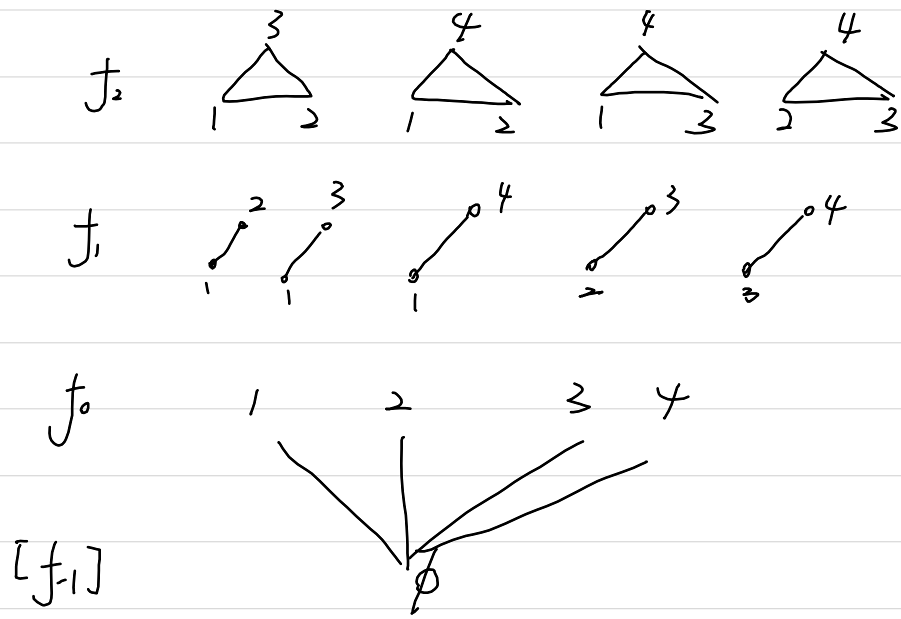
Let K be a simplicial complex.
Then its Hasse diagram is the (layered) graph

- consisting of the faces of K as its vertices.
- and edges whenever an $(i-1)$-face is contained in an i-face.

Ex:



S^2



tripartite graph

"Every edge is contained in exactly two triangles"

+

"Every triangle has exactly three edges"

Two ways to count the edge of the upper and layer of the Hasse diagram.

$$\left.\begin{array}{l} \uparrow : 2f_1 \\ \downarrow : 3f_2 \end{array}\right\} 2f_1 = 3f_2 \quad \text{double counting}$$

For surfaces:

- $\chi = n - f_1 + f_2$
- $2f_1 = 3f_2$

$\rightsquigarrow f_2 = \frac{2}{3} f_1$

$\rightsquigarrow \chi = n - f_1 + f_2 = n - f_1 + \frac{2}{3} f_1 = n - \frac{1}{3} f_1$

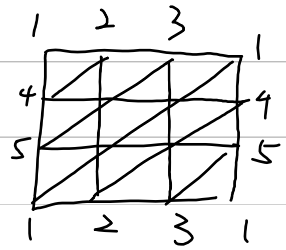
two equations for the three indeterminates n, f_1, f_2

$f_1 = 3n - 3\chi$

$f_2 = 2n - 2\chi$

i.e. $f = (n, 3n-3\chi, 2n-2\chi)$

Ex:



T^2, $\chi = 0$, $f = (9, 27, 18)$

Let K be a triangulation of a surface M with Euler classification $\chi(M)$ or n vertices. Then $f(K) = (n, 3n - 3\chi(M), 2n - 2\chi(M))$

$f = (n, 3n - 3\chi, 2n - 2\chi)$

$$f = (n+1, 3n - 3\chi + 3, 2n - 2\chi + 2)$$
$$= ((n+1), 3(n+1) - 3\chi, 2(n+1) - 2\chi)$$

Thus, if we know the minimal number n_{min} of vertices to triangulate a surface M, this completely determines the set of f-vectors of M.

Q: What kind of bounds do we know for n, f_1 or f_2?

A: A graph with n vertices can have at most $\binom{n}{2}$ edges.

i.e. $f_1 \leq \binom{n}{2}$ $\iff n^2 - 7n + 6\chi \geq 0$.

$$\underset{3n - 3\chi}{f_1} \leq \binom{n}{2} = \frac{n(n-1)}{2}$$

In case of equality:

$$n_{+/-} = \frac{7 \pm \overbrace{\sqrt{49-24\chi}}^{\geq 1}}{2} \longleftarrow \chi \leq 2$$

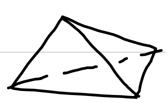
with $n_- = \dfrac{7-\sqrt{49-24\chi}}{2} \leq 3 \quad n_+ = \dfrac{7+\sqrt{49+24\chi}}{2}$

We can discard this solution since at least 4 vertices are needed to triangulate the surface.

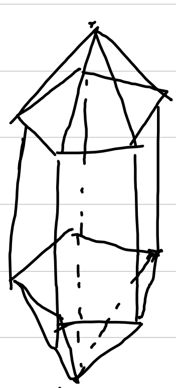
Thm [Heawood 1890] Heawood's bound.
Let M be a surface with Euler characteristic $\chi(M)$
Then a triangulation of M needs at least

$$M \geq \left\lceil \frac{7 \pm \sqrt{49-24\chi}}{2} \right\rceil \text{ vertices}$$

Q: How good is the bound?

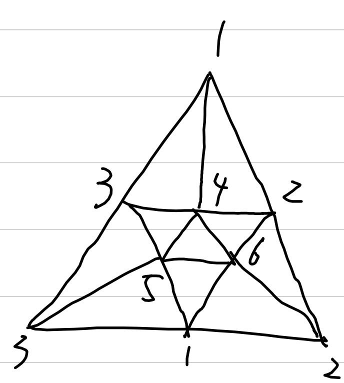
- $\chi = 2$: $n \geq 4$  S^2 $n=4$ ✓

- $\chi = 1$: $n \geq 6$



icosatedron
12 vertices

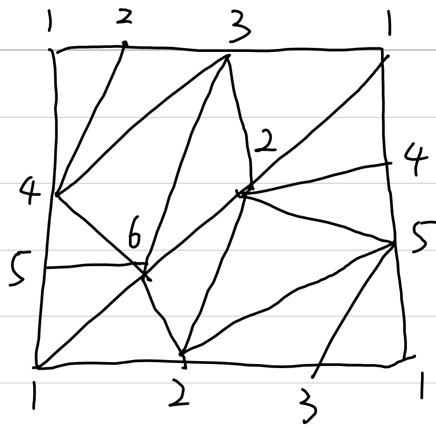
$\rightsquigarrow$ identification of antipodal points



$\mathbb{R}P^2_6$

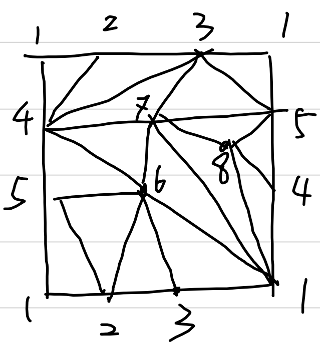
$n=6$

§ $\chi = 0$: $n \geq 7$



unique 7-vertex triangulation of the torus

Thm [Franklin 1934]
There is no 7-vertex triangulation of the Klein bottle.



Klein bottle with 8 vertices.

Alternative way to write Heawood's bound:

$$\binom{n}{2} \geq 3n - 3\chi = f_1$$

$$\Longleftrightarrow \frac{n \cdot (n-1)}{2} - 3n + 6 \geq 6 - 3\chi$$

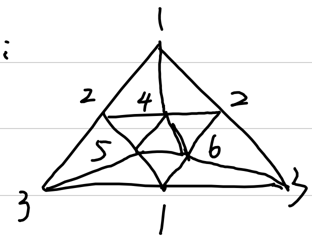
$$\Longleftrightarrow \frac{n^2 - 7n + 12}{2} \geq 3(2 - \chi)$$

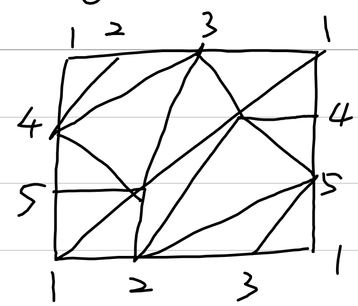
$$\Longleftrightarrow \binom{n-3}{2} \geq 3(2 - \chi)$$

with equality if and only if $f_1 = \binom{n}{2}$

Definition: A triangulated surface with $f_1 = \binom{n}{2}$ is called neighbourly.

Examples:





The edges of these triangulations form the complete graph K_n.

Thm (Vertex-minimal triangulations of surfaces)
[Reingel 1955: non-orientable surfaces]
[Yungman of Reingel 1980: orientable surfaces]

Let M be a surface $\neq$ orientable surface of genus 2, Klein bottle, non-orientable surface of genus 3

Then there is a triangulation of M with n vertices, if and only if

$$\binom{n-3}{2} \geq 3(2-\chi(M))$$

with equality if and only if the triangulation is neibourghly
(For the three exceptional cases $\binom{n-3}{2}$ has to be replaced by $\binom{n-4}{2}$, i.e. one extra vertex is needed [Humehe, 1978]

Week 6 Lecture 1

Characterization of the neighbourly triangulations of surfaces.

$$\begin{bmatrix} n-3 \\ 2 \end{bmatrix} = 3(2-\chi)$$

equality for $f_1 = \begin{bmatrix} n \\ 2 \end{bmatrix}$

$$\Leftrightarrow \quad \chi = 2 - \frac{1}{3}\begin{pmatrix} n-3 \\ 2 \end{pmatrix}$$

$$\Leftrightarrow \quad \underbrace{\chi}_{\text{integer}} = \underbrace{2}_{\text{integer}} - \underbrace{\frac{1}{3}\cdot\frac{(n-3)(n-4)}{6}}_{\text{integer}}$$

integer $\rightarrow$ integer

$n = 0, 1, 3, 4 \mod 6, \ n \geq 4$

$\Leftrightarrow n = 0, 1 \mod 3, \ n \geq 4$

For orientable surfaces: χ is even

$$\underbrace{\chi}_{\text{even}} = \underbrace{2}_{\text{even}} - \underbrace{\frac{(n-3)(n-4)}{6}}_{\text{even}} \qquad n = 0, 3, 4, 7 \mod 12, \ n \geq 4$$

Corollary (Neighborly triangulations)

Let M be a triangulated surface with n vertices.

The following are equivalent:

(i) M is neighborly

(ii) $\begin{bmatrix} n-3 \\ 2 \end{bmatrix} = 3(2-\chi(M))$

(iii) $\chi(M) = \dfrac{n(7-n)}{6}$

(iv) $n = \frac{1}{2}(7 + \sqrt{49 - 24\chi(M)})$

Existence: If M is a surface $\neq$ Klein bottle and n a number satisfying (ii) (iii) or (iv) then M has a neighborly triangulations with n vertices.

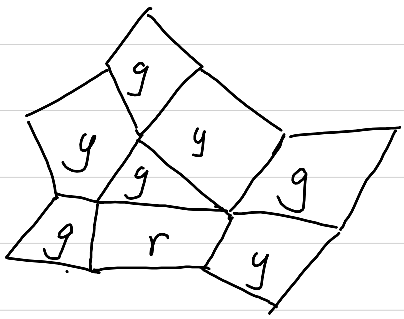
Literature: John Stillwell. Classical Topology and comb. Group Theory.

Wolfgang Kühnel. Tight Polyhedral Submanifolds and Tight triangulations.

Chapter 7: Map Coloring

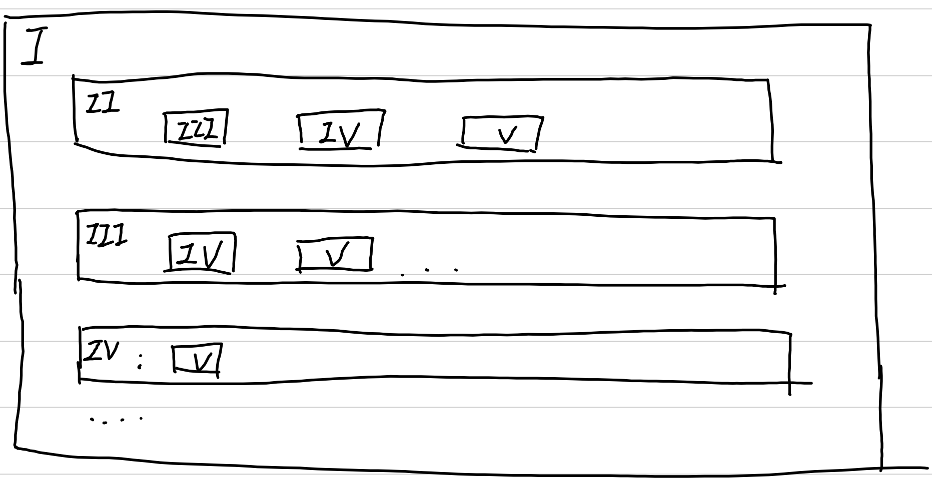
1852 Fracis Guthmé: "Four Color Problem"

"Can every map be colored with four colours?" (such that neighbooring countries have different colors)



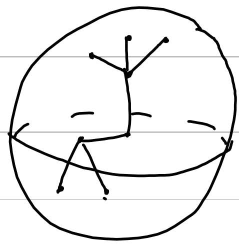
$\Rightarrow$ three colors.

Example:



First defintion: "Map" on a surface M = decomposition of M into a finite collection of cells (i.e. as a cell complex) consisting of

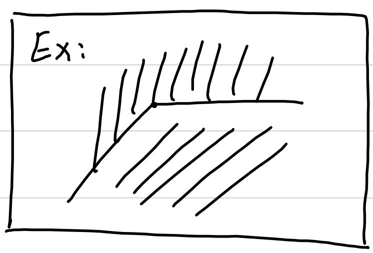
- vertices (0-cells)
- edges (1-cells)
- polygons (2-cells)

Ex: 

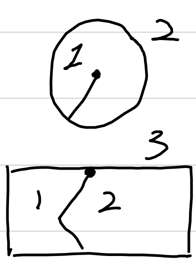
← decomposition of S^2 into 8 vertices, 7 edges, 1 14-polygon (with pairwise identified edges)

with $\chi = 8 - 7 + 1 = 2$

Refined definition: Map on a surface M
$=$ decomposition P of M into polygons
(a finite cell complex)
s.t. (i) $\deg(v) \geq 3$ for all vertices v.
(ii) every vertex with $\deg(v) = \varepsilon$ is incident with ε different polygons.

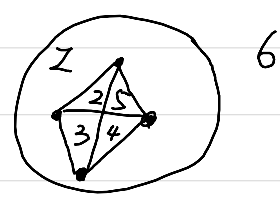


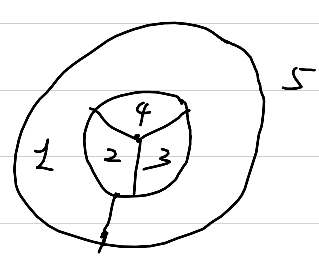
The polygons of a map are called countries (in particular they are connected)
Two countries are adjacent if they share an edge.
Excluded by the definition:



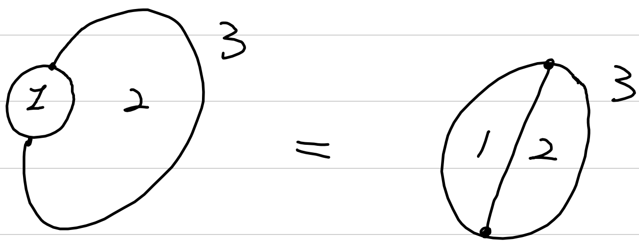
vertices of degree 1 or 2 can be removed to yield a simplified map.

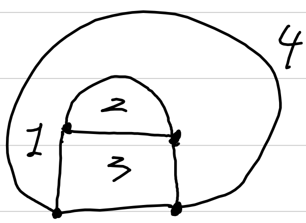
if a country touches itself, this splits the surface into independent parts
(for $M = S^2$)





O.E.:





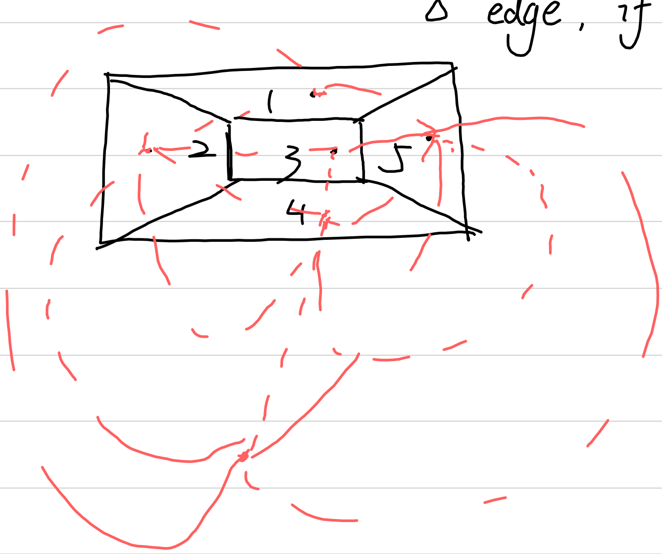
two countries can share multiple edges.

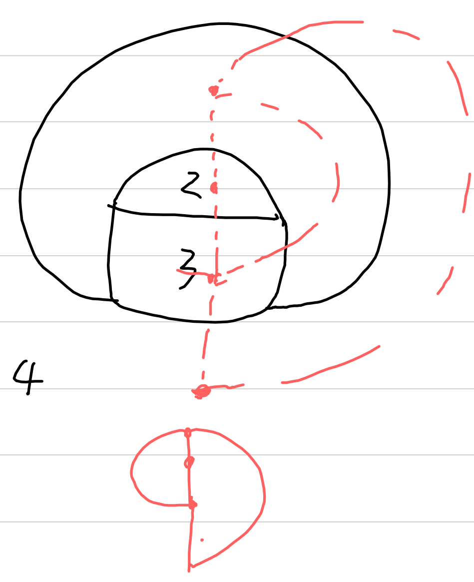
Chromatic number of a surface.

Let P be a map on a surface M

coloring of a map: coloring of the countries of a map. s.t. neighboring countries (that store on edge) have different colors.

country graph G_P: is the graph with
- node (vertex) for every country
- edge, if two countries are neighbors





Chromatic number of a map P

$\chi_{CH}(P) :=$ minimal number of colors that are needed to color P

Chromatic number of a surface

$$\chi_{CH}(M) := \max_{P \text{ map on } M} \chi_{CH}(P)$$

Q: Does the max exist?

A: Yes! (And it is $\leq$ norm for all surfaces M)

For the 2-sphere S^2: Four color theorem

[Appel, Haken, 1976]: computer proof with 1476 cases.

[Robertson, Sanders, Seymour, Thomas, 1996]: computer with 633 cases.

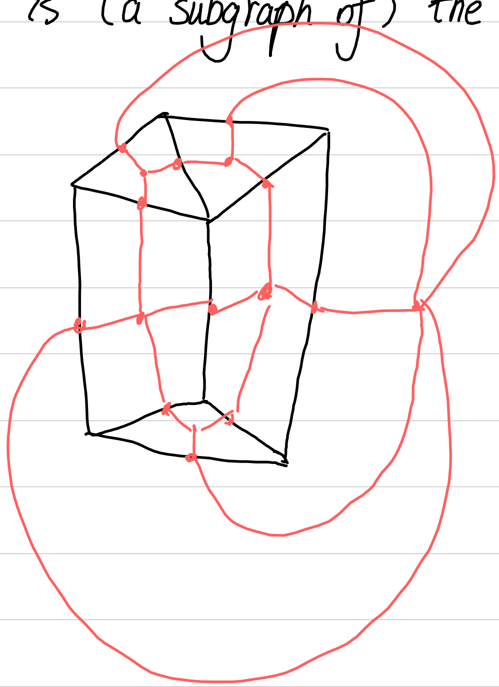
For surfaces $M \neq S^2$: Map Color Theorem [Ringel, Youngs, 1968]

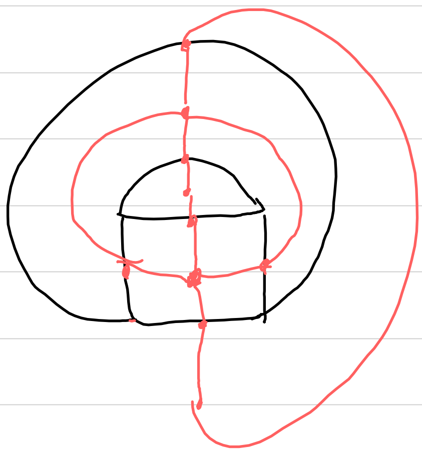
Aim: Proof of the Map Color Theorem $\rightsquigarrow$ several steps.

Thm: Let P be a map on a surface M. Then the country graph G_P can be embedded on M, i.e. $G_P \to M$.

Proof:
- place capitals inside countries
- subdivide edges by placing one subdivision node on each edge.
- connect capitals with the subdivision nodes of the adjacent edges.
- remove subdivision nodes.
- G_P is (a subgraph of) the resulting (multi-) graph.

Ex:





Remark: In case two neighbor countries of P always share exactly one edge, the resulting multi-graph is simple and isomorphic to G_P. In this case, the embedding of G_P on M defines the dual map to P.

Week 6 Lecture 2

Thm: Let G be a graph that can be embedded on a surface M, then there is a map P on M s.t. G is isomorphic to a subgraph of G_P

Proof:
- thicken the graph G on M
- cut and thickened edge
- the resulting pitchs arranged the nodes of G on M yield countries with country graph G.
- divide the rest of the surface into further countries (yielding G_P)

For every map P on M the country graph G_P can be embedded on M and $\chi_{CH}(P) = \chi_{CH}(G_P)$.

Thus: $\chi_{CH}(M) = \max\limits_{P \text{ map on } M} \chi_{CH}(P) = \max\limits_{P \text{ map on } M} \chi_{CH}(G_P) \leq \max\limits_{G \to M} \chi_{CH}(G)$

(under the assumption that the maxima exist, which we will prove by giving an upper bound on the right hand side)

If, conversely, $G \to M$, then there is a map P on M s.t. G is isomorphic to a subgraph of G_P. i.e. $\chi_{CH}(G) \leq \chi_{CH}(P)$.

Together: $\chi_{CH}(M) = \max\limits_{P \text{ map on } M} \chi_{CH}(P) = \max\limits_{G \to M} \chi_{CH}(G)$

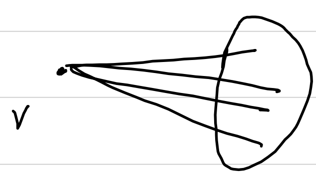
(still under the assumption that both maxima exist)

Some results from (topological) graph theory:

Def: A graph G is critical, if every proper subgraph has smaller chromatic number of G

Thm: If G is critical, then $\deg(v) \geq \chi_{CH}(G) - 1$ for all nodes v of G

Proof: Let v be a node with degree $< \chi_{CH}(G) - 1$.



$G \setminus v$ $\leq$ is colorable with $\chi_{CH}(G) - 1$ colors, since G is critical.

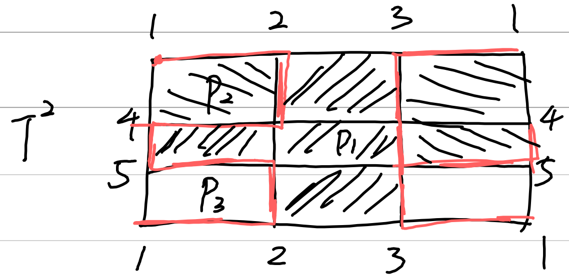
v has $\leq \chi_{CH}(G) - 2$ neighbors, i.e. one of the $\chi_{CH}(G) - 1$ colors is free for v.

Thm: Let G be a graph with α_0 vertices and α_1 edges and $\deg(v) \geq 2$ for every vertex v.

If G can be embedded on a surface M, then $\underline{\alpha_1 \leq 3\alpha_0 - 3\chi(M)}$

Euler characteristic

Proof: $G \to M$ then G is isomorphic to a subgraph G' of the 1-skeleton $\mathrm{skel}_1(P)$ of a map P, with $f(P) = (f_0, f_1, f_2)$
Let P be cut via G' into r partial polyhedra $P_1, P_2 \dots P_r$ with f-vectors $f^{(i)} = (f_0^{(i)}, f_1^{(i)}, f_2^{(i)})$ $i = 1, 2, \dots, r$.



G' $f^{(1)} = (8, 12, 4)$

edge

$f^{(2)} = (6, 8, 3)$

$f^{(3)} = (6, 7, 2)$

For P we have $f_2 = \sum_{i=1}^{r} f_2^{(i)}$

since every face of P. is contained in exactly one of the P_i

$\boxed{P_r}$

$f_1 = \sum_{i=1}^{r} f_1^{(i)} - \alpha_1'$ since the α_1' edges of G' once counted twice.

$f_0 = \sum_{i=1}^{r} f_0^{(i)} - \sum_{v \in G'} \deg(v) + \alpha_0'$

- a vertex v of P that is not in G' is counted exactly once in $\sum_{i=1}^{r} f_0^{(i)}$
- a vertex v of P that is in G' is connected $\deg(v)$ times in $\sum_{i=1}^{r} f_0^{(i)}$, and there are α_0' vertices in G'.

For G' we have by double counting: $\sum_{v \in G'} \deg(v) = 2\alpha_1'$

We then have: $\chi(M) = \chi(P) = f_0 - f_1 + f_2 = \sum_{i=1}^{r} (f_0^{(i)} - f_1^{(i)} + f_2^{(i)}) - 2\alpha_1' + \alpha_0' + \alpha_1'$

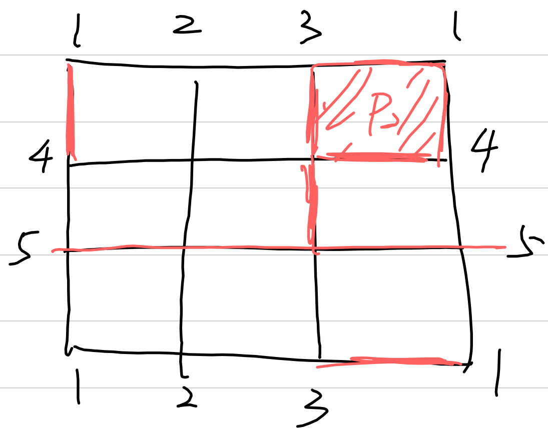
$= \sum_{i=1}^{r} \chi(P_i) - \alpha_1' + \alpha_0'$

$\leq r - \alpha_1' + \alpha_0'$

↑ for each partial polyhedron

as a corallary from $\chi(M) \leq 2$ ← $\chi(P_i) \leq 1$

Since G' has no node of $\deg \le 1$, each P_i has at least 3 edges.
Further every edge of G' lies in either one or two of the P_i:
Thus, $3r \le 2\alpha_1'$



Together,

$$3\chi(M) \le 3r - 3\alpha_1' + 3\alpha_0'$$
$$\le 3\alpha_0' - \alpha_1'$$

i.e. $\alpha_1' \le 3\alpha_0' - 3\chi(M)$

For the case of equality:

Thm: Let G be a graph with α_0 vertices and α_1 edges and $\deg(v) \ge 2$ for every vertex v. If G can be embedded on a surface M. and $\alpha_1 = 3\alpha_0 - 3\chi(M)$
then there is a map P on M s.t. G is isomorphic to $\text{skele}_1(P)$ and all faces of P are triangles.

with the converse:

Thm: If a graph G has a triangular embedding on a surface M, then $\alpha_1 = 3\alpha_0 - 3\chi(M)$

Aim: Show that $\max_{G \to M} \chi_{CH}(G)$ exists!

Let G be a graph embedded on a surface M, with $f_0 = n$ vertices and f_1 edges.

Then $f_1 \leq 3n - 3\chi(M)$ $(*)$

Thm: If G is critical then, $(\chi_{CH}(G) - 1)n \leq 2f_1$

Proof: For every vertex v of a critical graph we have $\deg(v) \geq \chi_{CH}(G) - 1$

Then $(\chi_{CH}(G) - 1)n \leq \sum_{i=1}^{n} \deg(v_i) = 2f_1$

double counting.

Next combine $(**)$ with $(*)$:

$$(\chi_{CH}(G) - 1)n \underset{(**)}{\leq} 2f_1 \leq 6n - 6\chi(M)$$

Thus: $\chi_{CH}(G) - 1 \leq 6 - \dfrac{6\chi(M)}{n}$ $(***)$

Week 7 Lecture 1

Last time: $\chi_{CH}(6) - 1 \le 6 - 6\frac{\chi(M)}{n}$ (***)

case I: $\chi(M) \le 0$

since $\chi_{CH}(6) \le n$, we obtain from (***):

$$\chi_{CH}(6)^2 - \chi_{CH}(6) \le 6\chi_{CH}(6) - 6\chi(M)$$

$$\Leftrightarrow \chi_{CH}(6)^2 - 7\chi_{CH}(6) + 6\chi(M) \le 0$$

$$\Leftrightarrow \underbrace{\left(\chi_{CH}(6) - \frac{7+\sqrt{49-24\chi(M)}}{2}\right)}_{\text{thus the first factor is } \le 0} \cdot \underbrace{\left(\chi_{CH}(6) - \frac{7-\sqrt{49-24\chi(M)}}{2}\right)}_{\ge 0} \le 0$$

since $\chi(M) \le 0$: $\sqrt{49-24\chi(M)} \ge 7$

since $\chi_{CH}(6) \ge 1$ the second factor is ≥ 0

i.e. $\chi_{CH}(6) \le \frac{7+\sqrt{49-24\chi(M)}}{2}$

Case 2: $\chi(M) = 1$ (projective plane $\mathbb{R}P^2$)

From (***): $\chi_{CH}(6) - 1 \le 6 - \frac{6}{n} < 6$

Since $\chi_{CH}(6)$ is an integer:

$$\chi_{CH}(6) \le 6 = \frac{7+5}{2} = \frac{7+\sqrt{49-24\chi(M)}}{2}$$

case III: $\chi(M) = 2$ (2-sphere S^2)

From (***): $\chi_{CH}(M) - 1 \le 6 - \frac{12}{n} < 6$

i.e. $\chi_{CH}(6) \le 6$

$\downarrow$

This is the upper-bound our recent proof can achieve.

Case I + II: $\chi(M) \leq 1$

Since then $\chi_{ch}(G) \leq \dfrac{7 + \sqrt{49 - 24\chi(M)}}{2}$

holds fore every critical graph embedded on M,
this bound holds for all graphs G
that can be embbeded on M and therefore
$\chi_{ch}(M) = \max\limits_{P \text{ map on } M} \chi_{ch}(P) = \max\limits_{G \to M} \chi_{ch}(G)$ is well-defined.

Then [Heawood, 1890]
Let M be a surface with $\chi(M) \neq 2$ (i.e. $M \neq S^2$),
then $\chi_{ch}(M) \leq \dfrac{7 + \sqrt{49 - 24\chi(M)}}{2}$ (Heawood's bound for colorings)

Thus (Six color Theorem)
$\chi_{ch}(S^2) \leq 6$

Then (Five color Theorem)
$\chi_{ch}(S^2) \leq 5$
i.e. every planar graph G can be colored with 5 colors.

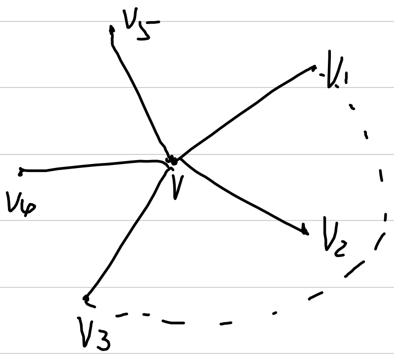
Proof: Let G be a planar connected graph with $n \geq 5$ vertices
By induction, we assume that every planar graph with fewer than n vertices can be colored with 5 colors.

For the average vertex-degree of G we have $d(G) = \dfrac{2f_1}{n} \leq \dfrac{2(3n-6)}{n} < 6$

So let $v \in V(G)$ be a vertex with $\deg(v) \leq 5$ then $H := G - v$ has by the hypothesis a 5-coloring $c: V(H) \to \{1, \ldots, 5\}$.

Case 1: The neighbors of v are colored with at most 4 colors.
we then can use the free color to color v.

Case 2: v has 5 neighbours that are colored differently.



For $i,j = \{1, 2, \ldots, 5\}$ let H_{ij} be the subgraph induced by the colors i and j.

Let C_1 be the component of H_{13} that contains v_1.

i) $v_3 \notin C_1$: Can swap colors 1 and 3 in component C_1.
Then we v_1 and v_3 have color 3 and v can be colored with 1.

ii) $v_3 \in C_1$: H_{13} contains some $v_1 - v_3$ path P
By the Jordan curve theorem the circle $v v_1 P v_3 v$ seperates v_2 and v_4 i.e. they lie in different components of H_{24}. we swap colors in one of the components.

Let $M \neq S^2$ be a surface:
Q: How good is Heawood's bound?
A: Best possible. except for the Klein Bottle.

Q. For what kind of critical graph?
A: complete graphs K_n.

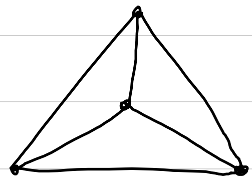
Thread problem / Faden problem
Given n. what is the smallest number $\gamma(n)$ of handles, so that on an (orientable) surface of genus $\gamma(n)$ there are n points that can pairwisely be connected

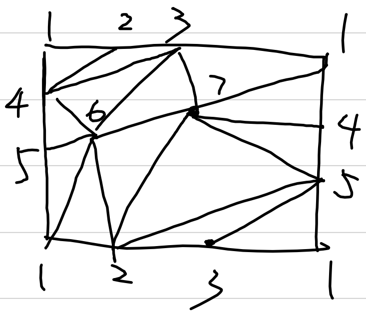
by curves that do not intersect each other

i.e. What is $\gamma(n)$ so that K_n embeds on the (orientable) surface of genus $\gamma(n)$?

Ex: $K_4 \to S^2$

$K_7 \to T^2$





Thm [Ringel, Youngs 1968]

$$\gamma(n) = \left[\frac{(n-3)(n-4)}{2} \right] \text{ for } n \geq 3$$

Then (Map Color Theorem) [Ringel Youngs 1968]

For any surfaces $M \neq S^2$

(and with the exception of the klein bottle) the following are equivalent:

(i) There is an embedding $K_n \hookrightarrow M$

(ii) $\chi(M) \leq \frac{n(7-n)}{6}$

(iii) $n \leq \frac{1}{2}(7 + \sqrt{49 - 24\chi(M)})$

(For the klein bottle (i) is equivalent to $n \leq 6$).

If equality holds in the above, then the embedding $K_n \hookrightarrow M$ defines a triangulation (actually, a neighbourly triangulation)

Two Heawood bounds

- vertex-minimal triangulations

$$n \geq \frac{7+\sqrt{49-24\chi(M)}}{2}$$

\# vertices

- colorings of surfaces (for $M \neq S^2$)

$$n \leq \frac{7+\sqrt{49-24\chi(M)}}{2}$$

$\chi_{cH}(M)$ for embeddings of the complete graph $K_n \hookrightarrow M$.

In case of equality:

$$n = \frac{7+\sqrt{49-24\chi(M)}}{2} \iff \chi(M) = \frac{n(7-n)}{6}$$

integer　　integer

We have neighborly triangulations with the cases

M orientable: $n \equiv 0, 3, 4, 7 \pmod{12}$

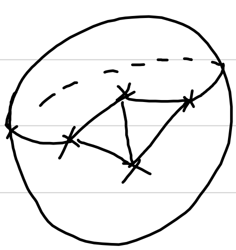
M non-orientable: $n \equiv 0, 1 \pmod 3$

Week 7 Lecture 2

How can we obtain (infinite series of) vertex-minimal triangulations?

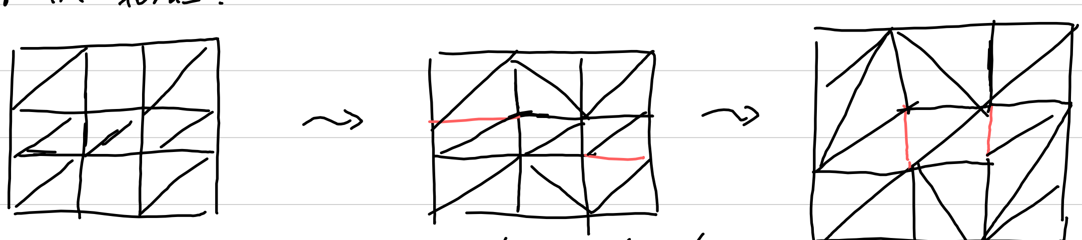
- by enumeration
- by local modification
- by construction

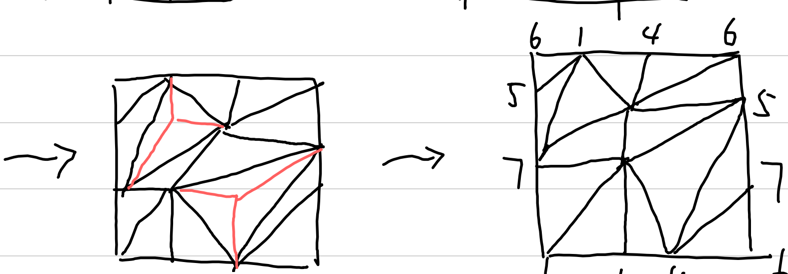
Bistellar flips are local modifications of a surface that do not change the topological type:



The new diagonal cannot be already be present in the triangulations

For the torus:





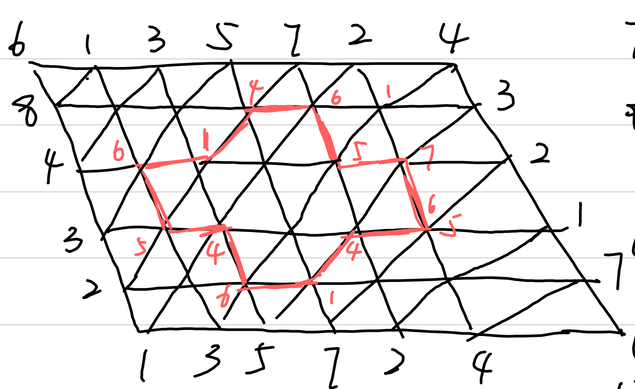
Mobius' 7-vertex torus.

Competition of the Paris Academy 1852

→ Perfectioner on quelque point important la théomé góometonique des polyédres «

Mobius contribution on surfaces and polyhedron from 1869 contains the 7-vertex triangulation of the torus:

The 7-vertex torus can be drawn on the triangulation grid with identification.



This triangulation is invariant under the cycle shift (1234567), e.g.

(1234567) · [1, 2, 4] = [2, 3, 5]

(1234567) · [2, 3, 5] = [3, 4, 6]

⋮

(The actual combinatorial automorphism group of the triangulation has size 42.

An infinite series of triangulations for $n \equiv 7 \mod 12$

Aim: Examples for which the vertex-stars are mapped onto each other by a cycle shift (This is possible if and only if $n \equiv 7 \mod 12$)

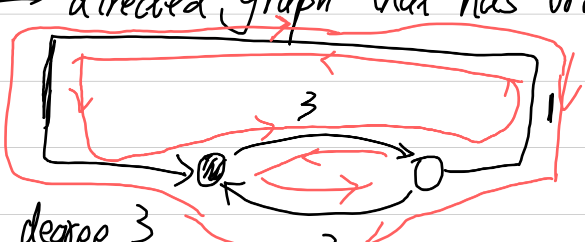
Construction principle

Start with a di-graph with oriented nodes

↪ directed graph that has oriented edges (arcs)

⊛ turn left

○ turn right

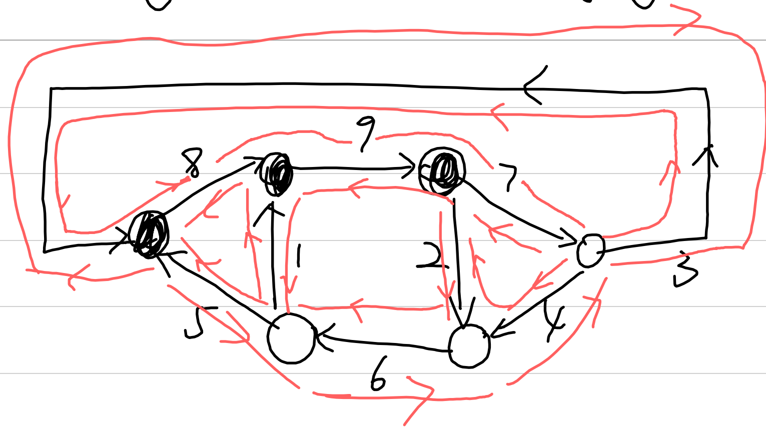


- Every node has degree 3.
- We obtain an induced cycle.
- Every element 1, 2, …, 63+3 of $\mathbb{Z}_{7+12 t}$ is used as flow capacity for one of the arcs.

△ At every node Kirchhoff's law holds:
the sum of incoming flows = sum of outgoing flows.

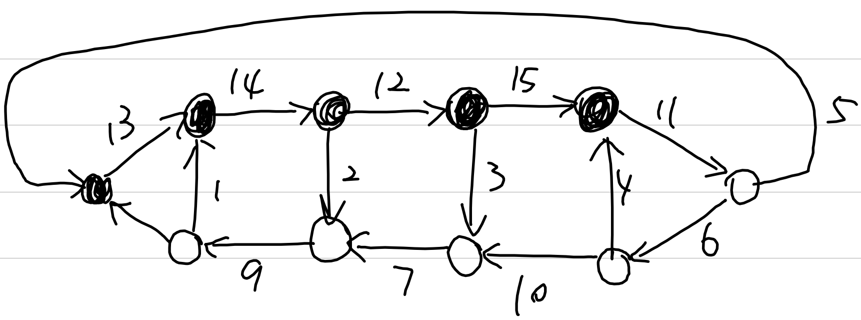
$n=19$

$\varepsilon=1$



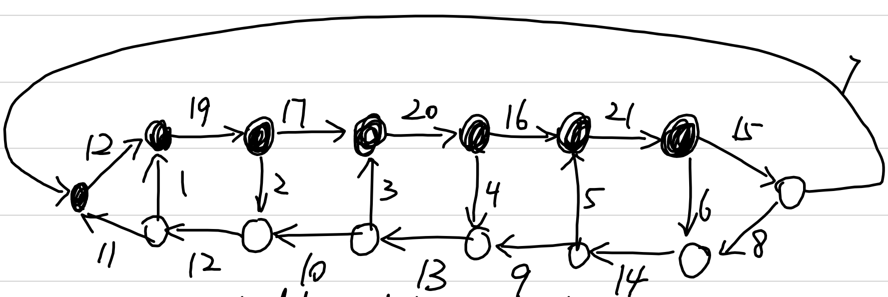
$n=31$

$\varepsilon=2$



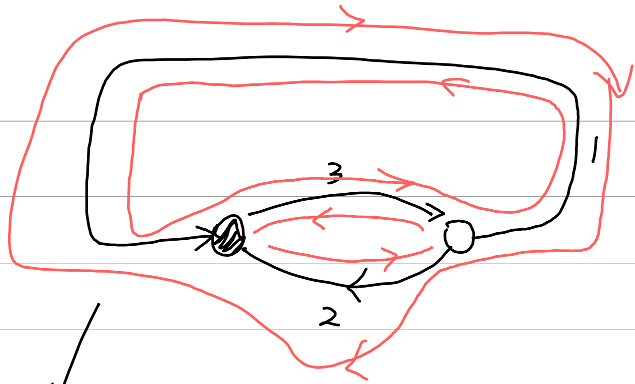
$n=43$

$\varepsilon=3$



ladder-like graph

Each of these diagrams records/encodes the star of vertex o for a triangulation surface via the arc labelings of the induced red circle.



For $n=7$:

$0: \; 1 \; 3 \; 2 \; \bar{1} \; \bar{3} \; \bar{2}$

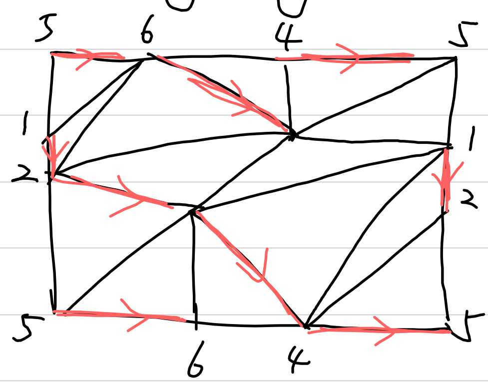
where $\bar{g}$ records a red arc has opposite direction w.r.t. the underlying black arc.
We interpret $\bar{g}$ as the inverse element of $g \in \mathbb{Z}_{7+12\varepsilon}$

We then obtain:

0 :	1	3	2	6	4	5
1 :	2	4	3	0	5	6
2 :	3	5	4	1	6	0
3 :	4	6	5	2	0	1
4 :	5	0	6	3	1	2
5 :	6	1	0	4	2	3
6 :	0	2	1	5	3	4

cycle stuff

with resulting triangulation:



Thm: This products on infinite series of $n \equiv 7 \mod 12$ vector triangulations.

Next simplest series:

uses $\mathbb{Z}_2 \times \mathbb{Z}_{2+6\varepsilon}$ as symmetry group.

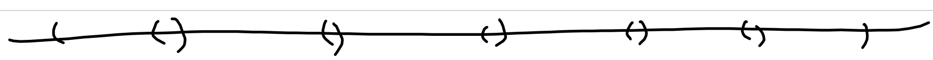
Week 8 Thursday

Def: Let X be a topological space and $\mathcal{U}: \{ U_i,\ i \in I \}$ a collection of open sets in X for some index set I.
Then the nerve (complex) $N(\mathcal{U})$ of $\mathcal{U}$. is the abstract simplicial complex constructed as follows:

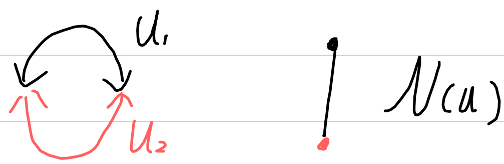
$$N(\mathcal{U}) = \{ y \subseteq I,\ y \text{ finite} : \bigcap_{j \in y} U_j \neq \phi \}$$

Remarks:

- $N(\mathcal{U})$ need not to be a finite simplicial complex E.g. consider $\mathbb{R} = \bigcup_{i \in I} (i - \varepsilon,\ i + 1 + \varepsilon)$ for $0 < \varepsilon < \frac{1}{2}$



- The nerve complex of a finite covering is a finite abstract simplicial complex.
- The topology of $|N(\mathcal{U})|$ can be different from the topologe of $X = \bigcup_{i \in I} U_i$
  E.g. $X = S^1$



with $N(\mathcal{U}) = \{ \phi, \{1\}, \{2\}, \{1,2\} \}$

History: The nerve construction goes back to Pavel Alexandvor
über der allgomeinen Dimensions-begriff und seine Bezietung
zu elementaren geometrischen Anschanung.
Math. Ann 98 (1928), 617-635.

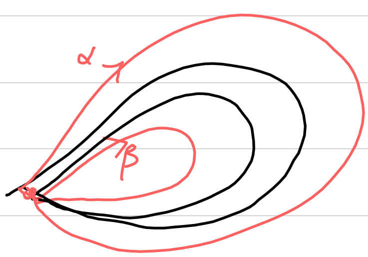
Remark: The nerve construction can be generalized to an arbitary family of sets $(X_i)_{i\in I}$, not necessarily open.

Aim: $\mathcal{N}(\mathcal{U})$ should capture the topology of $\mathcal{U}$.

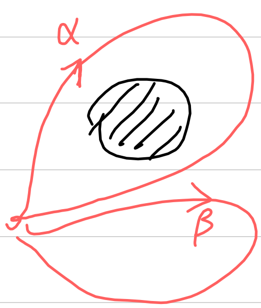
Def: Let $f, g: X \to Y$ be continuous maps. If there is a continuous map
$$F: X \times I \to Y$$
with $F(x,0) = f(x)$
$F(x,1) = g(x)$ for all $x \in X$.
then f is homotopic to g, $f \simeq_F g$
with F the homotopy between f and g.

If additionally $A \subseteq X$ and $F(a,t) = f(a) = g(a)$ for all $a \in A$, $t \in I$,
then F is a homotopy relative to A.
$f \simeq_F g$ rel A

Examples:

- 

continuous deformation of curves
$\alpha \simeq \beta$ rel pt.

- 

$\alpha \not\simeq \beta$ (rel pt)

Def: Let X and Y be topology space.
Then X and Y are homotopy equivalent, $X \simeq Y$, if there are continuous maps.

$$f: X \to Y \text{ and } g: Y \to X$$

s.t.

i) $f \circ g$ is homotopic to id_Y on Y,
$$f \circ g \simeq id_Y$$

ii) $g \circ f$ is homotopic to id_X on X.
$$g \circ f \simeq id_X$$

Example: 

Def: A covering $\mathcal{U} = \{U_i\}_{i \in I}$ is a good cover if every $\sigma \subseteq I$ the intersection $\bigcap_{i \in \sigma} U_i$ is either
- empty or
- contractible

Thm: (Nerve Lemma / Nerve Theorem)
Let X be a topology space with a good covering $X = \bigcup_{i \in I} U_i$
then $N(\mathcal{U}) \simeq X$

Remark: The intersection of (finitely) many convex sets U_i in Euclidean space is either empty or convex (and thus contractible). Therefore a covering by open/closed dishes is good.

Version for simplicial complexes.

Let $K_1, \dots, K_n$ be subcomplexes of a finite simplicial complex K and let $A_i = |K_i|$

If the intersection $\bigcap_{j \in \mathcal{Y}} A_j$ is either empty or contractible for every subset $\mathcal{Y} \in \{1, \dots, n\}$

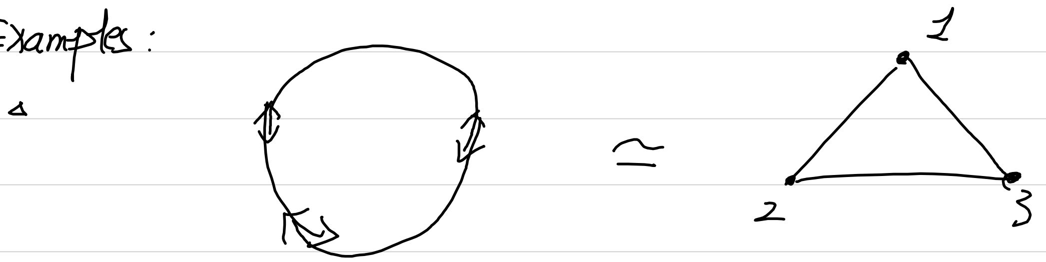
then $|N(\{A_1, A_2, \dots, A_n\})| \simeq |K|$, i.e. the nerve $N(A)$ is homotopy equivalent to K.

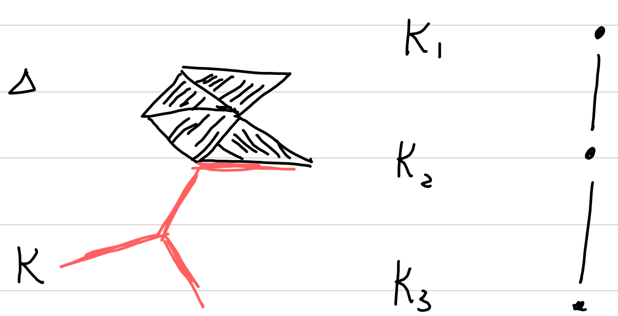
History: Karol Borsuk

On the embedding of systems of compacta in simplicial complexes.

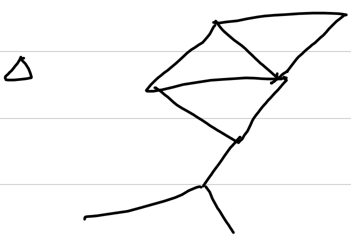
Fund. Math 35 (1948), 217-234.

Examples:





$N(\{K_1, K_2, K_3\})$



$K_1 \simeq \bullet \quad N(\{K_1\}) = pt$

Def: Let K be a simplicial complex. The standard covering of K is the covering $\mathcal{F} = (F_j)_{j\in \mathcal{Y}}$ of K by its facets F_j, $j \in \mathcal{Y}$. We write $\mathcal{N}(K)$ for the nerve complex $\mathcal{N}(\mathcal{F})$ of the standard covering $\mathcal{F}$ of K.

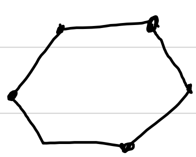
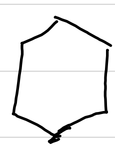
Examples:

- $\mathcal{N}($  $) =$ 

$\mathcal{N}(\mathcal{N}($  $)) = \mathcal{N}($  $) =$ 

$\vdots$

$\mathcal{N}^4($  $) =$

$\mathcal{N}($  $) =$ 

Week 8 Lecture 1

Taut : every vertex is the intersection of facets containing it.

Lemma: If K is taut, then $\mathcal{N}(K)$ is taut as well and

$$K = \mathcal{N}(\mathcal{N}(K))$$

On the other hand, let $j \in y$ be a facet index.

Then $j \in \bigcap_{v \in V, j \in \Delta_v} \Delta_v$

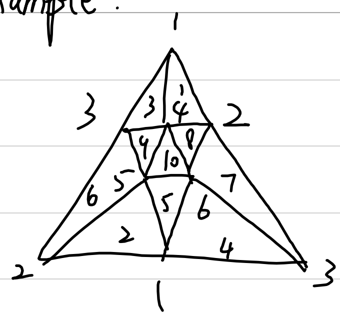
Suppose there is some $j' \in y$, $j' \neq j$

s.t. $j' \in \bigcap_{\underbrace{v \in V, j \in \Delta_v}_{\hookrightarrow v \in F_j}} \Delta_v$.

then $F_j \subsetneq F_{j'}$, which is a contradiction to the maximality of F_j.

Therefore $\bigcap_{v \in V, j \in \Delta_v} \Delta_v = \{j\}$.

Example:



$\mathbb{R}P^2_6$

1	[1, 2, 4]	6	[2, 3, 5]
2	[1, 2, 5]	7	[2, 3, 6]
3	[1, 3, 4]	8	[2, 4, 6]
4	[1, 3, 6]	9	[3, 4, 5]
5	[2, 3, 5]	10	[4, 5, 6]

$N(\mathbb{R}P^2)$

1 [ ] 2 [ ]

is a 2-dimensional simplicial complex on 10 vertexs.

Chapter 9 Čech complex.

Def (Čech complex)

For $S \subseteq X$ and $r > 0$ let $(B_r(S_i))_{i \in I}$ be the collection of closed balls with radius r arround the points $S_i \in S$.

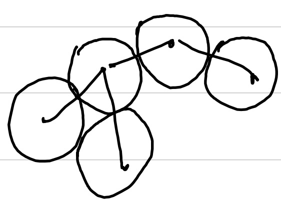
points set.

Then,

$$\check{C}ech_r(S) = \{ \mathfrak{z} \subseteq S \mid \bigcap_{i \in \mathfrak{z}} B_r(S_i) \neq \phi \}.$$

is the Čech complex for S and given r.

Example:



Remarks:

(1) $\check{C}ech_r(S)$ is

(2) $\bigcap_{i \in \mathfrak{z}} B_r(S_i) \neq \phi$ iff $\mathfrak{z} \subseteq S$ lies in a ball of radius r

"⇒" Let $x \in \bigcap_{i \in \mathfrak{z}} B_r(S_i) \neq \phi$,

then $\| S_i - x \| \leq r$ for all $S_i \in \mathfrak{z}$

i.e. $S_i \in B_r(x)$ for all $S_i \in \mathfrak{z}$.

"⇐" (We use $\| S_i - x \|$ in Euclidean space and $d(S_i, x)$ in a general metric space)

(3) $\check{C}ech_{r_1}(S) \leq \check{C}ech_{r_2}(S)$ for $r_1 \leq r_2$.

The miniball algorithm is a randomized recursive procedure.

Week 9 Lecture 1

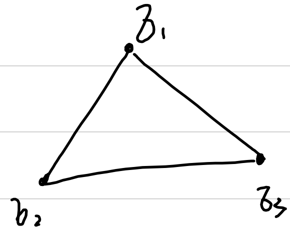
Chapter 10 Vietoris - Rips complex.

Leopold Vietoris: June 4, 1891 - April 9, 2002

was oldest living person in Austria 110 years and 10 months

Let $S \subseteq \mathbb{R}^d$ be a finite set and $\sigma \subseteq S$. Then the diameter of σ is

$$\operatorname{diam} \sigma := \max\{ |\sigma_i - \sigma_j| \mid \sigma_i, \sigma_j \in \sigma \}$$



longest edge.

Def: Let $S \subseteq \mathbb{R}^d$ and $r > 0$

The Vietoris - Rips complex of S with r is

$$VR_r(S) := \{ \sigma \subseteq S \mid \operatorname{diam} \sigma \leq 2r \}.$$

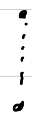
Computation of $VR_r(S)$

For $n = |S|$ compute all $\binom{n}{2}$ distances for pairs of vertices, and then add 2- and higher-dim, simplices whenever $\operatorname{diam} \sigma \leq 2r$.

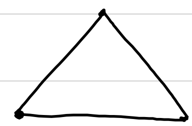
Remark: For $r >> 0$; $VR_r(S) = \Delta_{|S|-1}$ is high-dimensional with $2^{|S|}$ faces.

Def: Let K be an abstract simplicial complex. A subset $\sigma \subseteq \mathrm{Vert}(K)$, $|\sigma| \geq 2$, is a minimal non-face / empty face / missing face of σ if $\partial\sigma < K$. but $\sigma \notin K$.

Examples:



missing edge



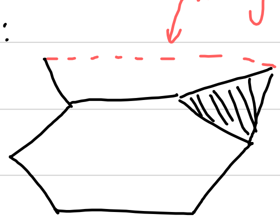
missing triangle.

Def (flag complexes)

K is flag, if the minimal non-faces of K have only two elements.
i.e if $\sigma \subseteq \mathrm{Vert}(K)$, $|\sigma| \geq 3$, with $\partial\sigma < K$, then $\sigma \in K$.

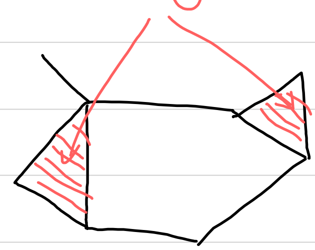
Examples:

non-face



flag

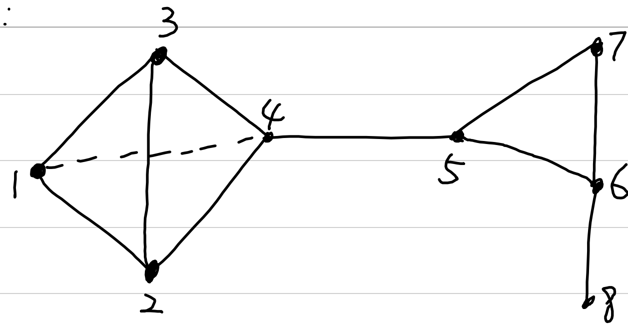
non-faces that are triangles.



Def: Let $G = (V, E)$ be a graph. A clique of G is a complete subgraph of G.

Def: The clique complex (CG) of a graph $G = (V, E)$ is the abstract simplicial complex on V consisting of all cliques of G.

Example:



facets of $C(G)$: $[1,2,3,4]$, $[4,5]$, $[5,6,7]$, $[6,8]$.

Lemma: The clique complex $C(G)$ of a graph G is flag.

Lemma: Vietoris - Rips complexes are flag

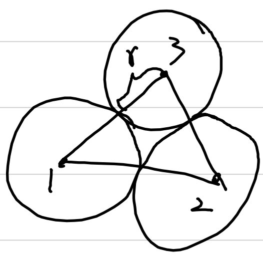
Remark: All information or a flag complex is already in its 1-skeleton.

comparison: $\check{C}ech_r(S) = \{ \sigma \subseteq S \mid \bigcap_{i \in \sigma} B_r(S_i) \neq \emptyset \}$

$VR_r(S) = \{ \sigma \subseteq S \mid diam \leq 2r \}$

Lemma: $\check{C}ech_r(S) \leq VR_r(S)$

Example:

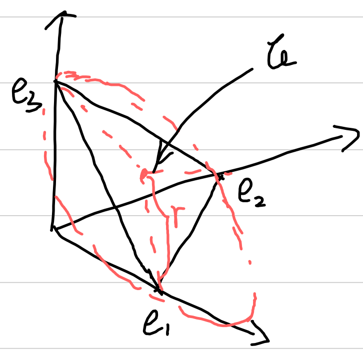


missing triangle in $\check{C}ech_r(S)$, but included in $VR_r(S)$ in $VR_r(S)$.

Lemma (Vietoris-Rips Lemma)

$$VR_r(S) \subseteq \check{C}ech_{\sqrt{2}r}(S)$$

Proof: Consider the standard d-simplex Δ_d in $\mathbb{R}^{d+1}$

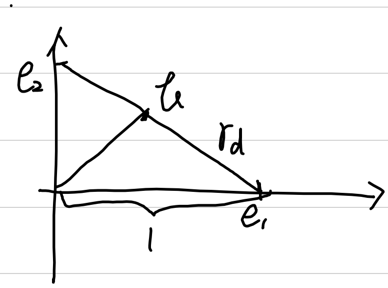


- Δ_d has edges of length $\sqrt{2}$
- barycenter has coordinates.

$$C_e = \left.\begin{bmatrix} \frac{1}{d+1} \\ \vdots \\ \frac{1}{d+1} \end{bmatrix}\right\} d+1$$

- $\|C_e\| = \underbrace{\sqrt{(\frac{1}{d+1})^2 + \dots + (\frac{1}{d+1})^2}}_{d+1} = \frac{1}{\sqrt{d+1}}$

- the smallest d-sphere that encloses Δ_d is centered at C_e and has radius r_d.



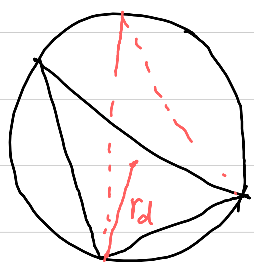
$\|b\|$

with $r_d^2 = 1 - \|C_e\|^2 = \frac{d}{d+1}$

i.e. $r_d = \sqrt{\frac{d}{d+1}}$ with $r_d \leq 1$ and

$r_d \to 1 \ (d \to \infty)$

$\triangle$ any set of $d+1$ or fewer points for which a d-ball of radius r_d is the smallest enclosing ball, has a pair of points of distance $\sqrt{2}$ or larger.



$\triangle$ every σ with $\operatorname{diam} \sigma \le \sqrt{2}$ belongs to $\check{C}ech_{r_d}(S)$

$\triangle$ by multiplying with $\sqrt{2} r$:

$$VR_r(S) \le \check{C}ech_{\sqrt{2}r \cdot r_d}(S)$$
$$\le \check{C}ech_{\sqrt{2}r}(S)$$

since $r_d \le 1$ for all d. $\square$

Together: $\check{C}ech_r(S) \le VR_r(S) \le \check{C}ech_{\sqrt{2}r}(S)$.

$\triangle$ $\check{C}ech_r(S)$ is a nerve complex.

$\triangle$ $VR_r(S)$ is flag and is the clique complex of the 1-skeleton of $\check{C}ech_r(S)$

i.e.

$$VR_r(S) = C(\operatorname{skel}_1(\check{C}ech_r(S)))$$

i.e. empty faces of $\check{C}ech_r(S)$ are filled in to yield $VR_r(S)$.

$\triangle$ By the Vietoris-Rips lemma, both complex types roughly contain the same topological information on the set S.

△ Computational bottle neck:

For $r \gg 0$, both complex types become high-dimensional with up to $2^{|S|}$ faces, which makes it infeasible to.

- set up the complexes.
- and also to further compute topological invariants.

Week 9 Lecture 2

Chapter 11: Voronoi diagrams and Delunung triangulations

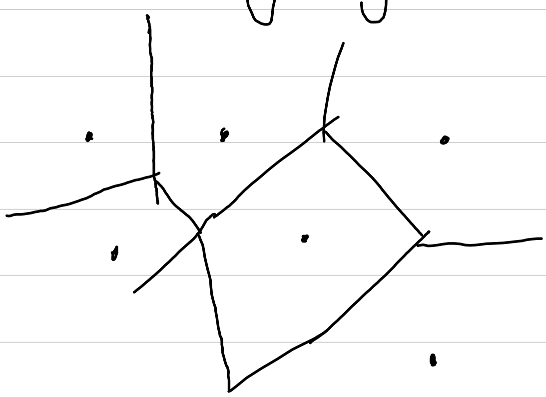
Let $S \subseteq \mathbb{R}^d$ be a (finite) set of points

Def: The Voronoi cell of point $S_i \in S$ is the set of points $x \in \mathbb{R}^d$ cosest to S_i:

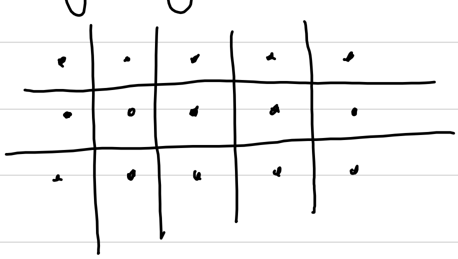
$$V_{S_i} = \{ x \in \mathbb{R}^d \mid \|x - S_i\| \leq \|x - S_j\|,\ S_j \in S \}$$

The Voronoi diagram of S is the collection of Voronoi cells of its points.

Ex:

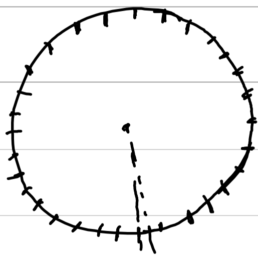


Ex: Voronoi diagram of a countable set of points.



Def: The Delaunly complex of S is the nerve complex of the Voronoi diagram of S.

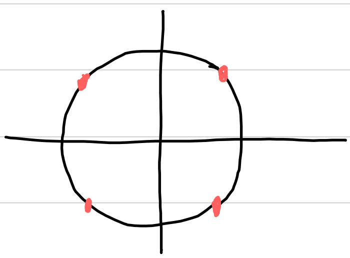
$$Del(S) = \{ \sigma \subseteq S \mid \bigcap_{S_i \in \sigma} V_{S_i} \neq \phi \}.$$

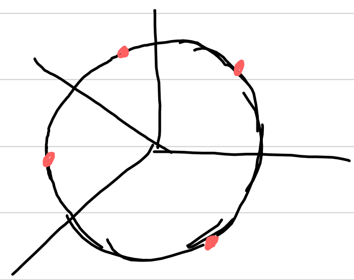


Ex:

Delaunay triangulation dual to the voronoi diagram.

△ Four points that lie on a common circle



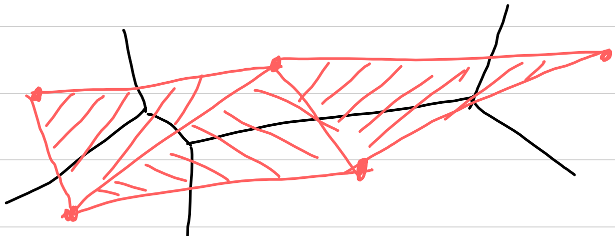


Def: The set S is in general position if no $d+2$ of its points lie on a common $(d-1)$-sphere.

For $d=2$: No for points on a circle (or on a line)

Lemma: If S is in general position, then $\dim(\mathrm{Del}(S)) \leq d$.

Def: If S is in general position, then the Delaunay triangulation of S is the geometric realization of $\mathrm{Del}(S)$ as the geometric simplicial complex on the points $S \in \mathbb{R}^d$ consisting of the simplices of $\mathrm{Del}(S)$.



Remark: For a finite set $S \subseteq \mathbb{R}^d$ in general position, the Delaunay triangulation of S is a triangulation of the convex hull $\operatorname{conv}(S)$ of S.

Weighted Voronoi diagrams.

Modified models to describe varying strength of influence.

Addictively weighted Voronoi diagrams.

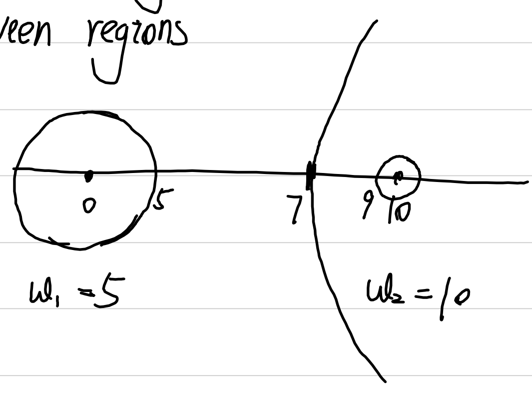
$$d_{s_i}(x) = \| x - s_i \| - w_i \text{ for positive weights } w_i.$$

For points on the boundary of two regions we have $d_{s_i}(x) = d_{s_j}(x)$.

$$\Leftrightarrow \| x - s_i \| - \| x - s_j \| = w_i - w_j.$$

$\rightsquigarrow$ For $w_i - w_j \neq 0$ this is the equation of a hyperbola as "bisector" between regions

Ex:



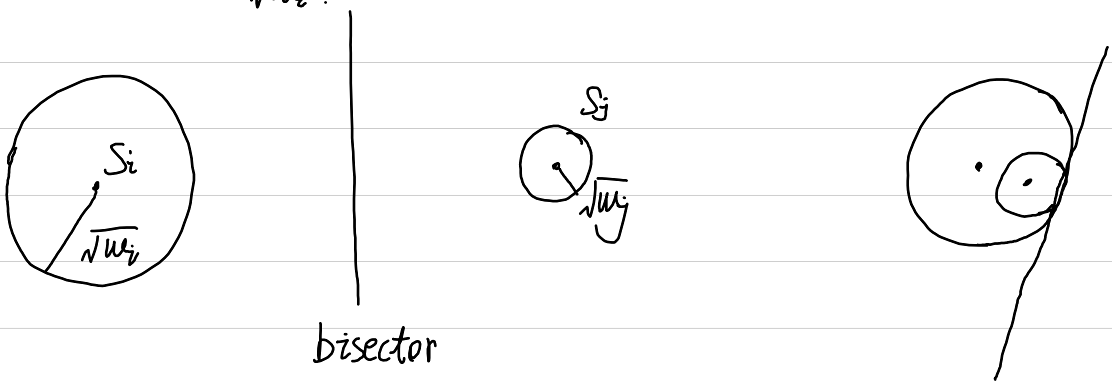
$$\left\| \begin{pmatrix} 7 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\| - \left\| \begin{pmatrix} 7 \\ 0 \end{pmatrix} - \begin{pmatrix} 10 \\ 0 \end{pmatrix} \right\| = 4$$
$$= 5 - 1$$

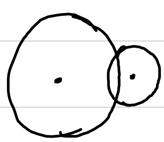
Power diagram or Laguerre - Voronoi diagram

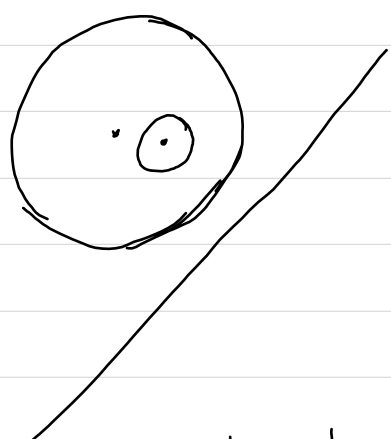
$$d_{S_i}(X) = \|X - S_i\|^2 - w_i \quad \text{for positive weights } w_i.$$

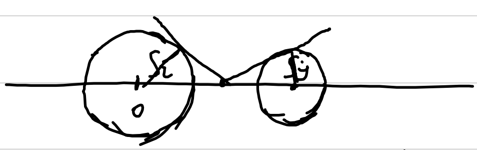
power distance

Examples: The weighted point S_i can be interpreted as a d-ball with radius $\sqrt{w_i}$.

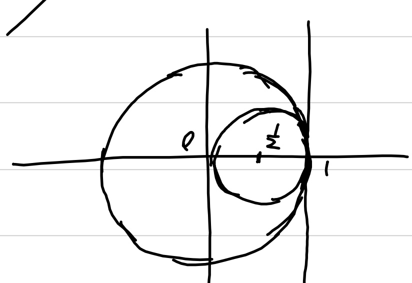








$\sqrt{w_i} = 1 \qquad \sqrt{w_j} = \frac{1}{2}$



$$\|X - 0\|^2 - 1 = \|X - 2\|^2 - \frac{1}{4}.$$

$$\iff X^2 - 1 = (X-2)^2 - \frac{1}{4}$$

$$\iff 4X = \frac{19}{4}$$

$$\iff X = \frac{19}{16}$$

$$\|X - 0\|^2 - 1 = \|X - \tfrac{1}{2}\|^2 - \frac{1}{4}.$$

Multiplicatively weighted Voronoi diagrams.

$$d_{S_i}(X) = \frac{\| X - S_i \|}{w_i}$$

Δ bisectors are cicular arcs (or line segments)

Def: The (weighted) Voronoi cell of a point $S_i \in S$ is the set of points $x \in \mathbb{R}^d$ closest to S_i:

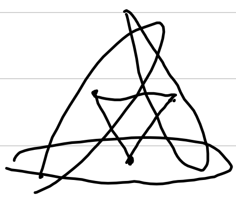
$$V_{S_i} = \{ x \in \mathbb{R}^d \mid d_{S_i}(x) \leq d_{S_j}(x),\ S_j \in S \}$$

The (weighted) Voronoi diagram of S is the collection of Voronoi cells of its points.

Def: The (weighted) Delaunay complex of S is the nerve complex of the (weighted) Voronoi diagram of S.

Week 10 Lecture 1

Nerve complex: an abstract complex that records the pattern of intersections between the sets in the family



Alpha Complexes

- nested family of subcomplexes of the Delaunay complex
- similar to Čech complexes but
  - bounded in dimension
  - and with canonical realization.

if S is in general position.

Chapter 12: Simplicial Homology

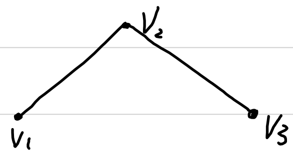
Recall: For a finite set of points $S \subseteq \mathbb{R}^d$ we want to compute invariants for associated simplicial complexes:

$$\begin{array}{ccc} S & \longrightarrow & K(S) \\ & & \downarrow \\ & & I(K(S)) \end{array}$$

I: {simplicial complexes} $\to \mathbb{R}$
numerical invariant, e.g. Euler characteristic.

I: {simplicial complexes} $\to$ {Groups}
algebraic invariant, e.g. homology, fundamental group.

Def: $C_i(K, R) :=$ { formal linear combinations of i-faces of K with coefficients in R }
is the <u>i-th chain module</u>

Example: $K =$ 

$$C_0 = \{\alpha \cdot \underbrace{v_1}_{\{v_1\}} + \beta v_2 + \gamma v_3 \mid \alpha, \beta, \gamma \in \mathbb{R}\}$$

$$C_1 = \{\delta \cdot \underbrace{v_1 v_2}_{\{v_1, v_2\}} + \varepsilon v_2 \cdot v_3 \mid \delta, \varepsilon \in \mathbb{R}\}$$

The i-faces form an R-basis of $C_i(K, R)$.

Hence, letting

$$\underbrace{\partial_i (v_0 v_1 \dots v_i)}_{\text{ordered } i\text{-simplex}} := \sum_{j=0}^{i} (-1)^j v_0 \dots v_{j-1} v_{j+1} \dots v_i$$

$$= \sum_{j=0}^{i} (-1)^j v_0 \dots v_{j-1} \hat{v}_j v_{j+1} \dots v_i$$

$\hat{v}_j$ is "omitted"

$\partial_2 (v_0 v_1 v_2) = (-1)^0 v_1 v_2 + (-1)^1 v_0 v_2 + (-1)^2 v_0 v_1$

defines an R-linear map from $C_i(K, R)$ to $C_{i-1}(K, R)$

i.e.

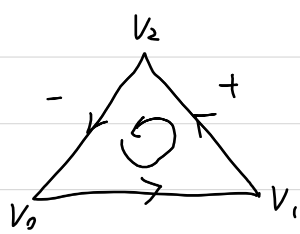
$$\partial_i : C_i(K, R) \to C_{i-1}(K, R)$$

Notation: we call ∂_i the i-th boundary map of K_i, $i \geq 1$

Particularly interesting:

- $R =$ field $\rightsquigarrow$ $C_i(K, R)$ is a vector space
- $R = \mathbb{Z}_2$ $\rightsquigarrow$ widely used in computational topology
- $R = \mathbb{Z}$ $\rightsquigarrow$ this is the universal case (the universal coefficient Theorem from topology tells us that results for other rings can be deduced from the $\mathbb{Z}$ case)

Example: Let $\sigma = v_0 v_1 v_2$ $[= \{v_0, v_1, v_2\}]$ be an ordered triangle.



$\partial_2(\sigma) = \partial_2 (v_0 v_1 v_2) = v_1 v_2 - v_0 v_2 + v_0 v_1$

$(\partial_1 \circ \partial_2)(\sigma) = \partial_1(\partial_2(v_0 v_1 v_2)) = \partial_1 (v_1 v_2 - v_0 v_2 + v_0 v_1)$

$=$

Lemma: $\partial^2 = 0$, i.e. $\triangle\ \partial \circ \partial = 0$

$\triangle\ \partial_{i-1} \circ \partial_i = 0$

Proof: $\partial^2(v_0 \dots v_{i+1}) = \partial\left(\sum_{j=0}^{i+1} (-1)^j v_0 \dots \hat{v}_j \dots v_{i+1}\right)$

$$= \sum_{j=0}^{i+1} (-1)^j \partial(v_0 \dots \hat{v}_j \dots v_{i+1})$$

$$= \sum_{j=0}^{i+1} (-1)^j \sum_{i=0}^{j-1} (-1)^i (v_0 \dots$$

Def: $B_i(K,R) :=$ im ∂_{i+1}, $\quad \partial_{i+1} : C_{i+1}(K,R) \to C_i(K,R)$

$Z_i(K,R) := \quad \partial_i \qquad \partial_i : C_i(K,R) \to C_{i+1}(K,R)$

$\left.\begin{array}{l} B_i(K,R) \text{ i-th boundary module} \\ Z_i(K,R) \text{ i-th module of cycles} \end{array}\right\}$ (free) R-

Def: $H_i(K,R) := Z_i(K,R)/B_i(K,R)$

quotient module.

is the i-th homology module (i-th homology group)

Remark: Each finitely generated abelian group G has two uniquely determined subgroups F and T, where F is free and T is finite.

Def: $r_i :=$ free rank of $H_i(K,\mathbb{Z})$, i-th Betti number of K.

Example: $H_i(K,\mathbb{Z}) \cong \underset{F}{\mathbb{Z}}^{\underset{\beta_i}{w}} \oplus \underbrace{\mathbb{Z}_2 \oplus \mathbb{Z}_5}_{T}$

Theorem: $\sum_{i=0}^{d} (-1)^i f_i(K) = \chi(K) = \sum_{i=0}^{d} \beta_i$

Notation: For $C \in Z_i(K,R)$

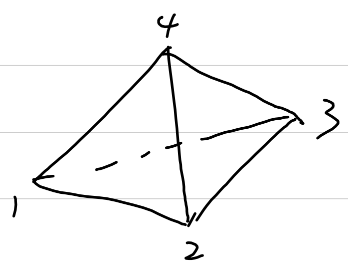
let $[c] := C + B_i(K,R) \in H_i(K,R)$

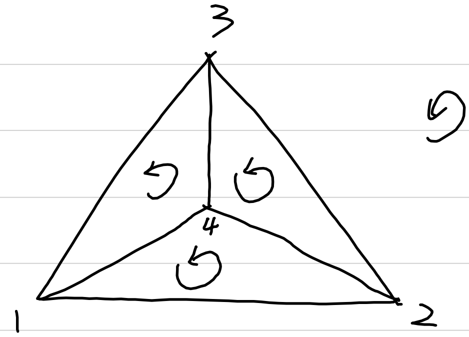
be the homology class of C

Example: $\partial \Delta_3 \cong S^2$

boundary ↗ ↖ 3-simplex = teturahedron

schlegel diagram





$$\partial_3(1234) = 234 - 134 + 124 - 123$$
$$= -123 + 124 - 134 + 234$$

compute H_2:

$$\begin{array}{c|cccc}
\partial_2 & 123 & 124 & 134 & 234 \\
\hline
12 & 1 & 1 & 0 & 0 \\
13 & -1 & 0 & 1 & 0 \\
14 & 0 & -1 & -1 & 0 \\
\hline
23 & 1 & 0 & 0 & 1 \\
24 & 0 & 1 & 0 & -1 \\
34 & 0 & 0 & 1 & 1
\end{array}$$

$+13 + 14$

$+14 + 23 + 24$

$+24 + 34$

} can be eliminated

$\leadsto$ $\boxed{\operatorname{Ker} \partial_2 = \left\{ s \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; s \in \mathbb{Z} \right\}}$ $\cong \mathbb{Z}$

$$\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \begin{array}{l} \leftarrow -123 \\ \leftarrow +124 \\ \leftarrow -134 \\ \leftarrow +234 \end{array}$$

$B_2 = \mathrm{im}\, \partial_3 = 0$

$\rightsquigarrow H_2 = \langle [-123 + 124 - 134 + 234] \rangle$

$= \langle (-123 + 124 - 134 + 234) + \underbrace{B_2}_{0} \rangle \cong \mathbb{Z}$

i.e. $H_2 \cong \mathbb{Z}$

Compute H_1:

The i-th column in the boundary matrix ∂_2

Chapter 13 Smith Normal Form (SNF)

For case R is a field, then in ∂_{i+1} and ∂_i and thus H_i can be computed via Gaussian elimination

But, Gaussian elemination might fail for $R = \mathbb{Z}$ (in case multiplication inverses are needed for the elimination)

way out: Use Lemma of Bézout instead of multiplicative inverses.

Lemma of Bézout.

$\forall a, b \in \mathbb{Z} : \exists s, t \in \mathbb{Z} : \gcd(a, b) = s \cdot a + t \cdot b.$

Def: Let R be a commutative ring with $1 \neq 0$

Then R is a <u>principal ideal domain</u> (PID)

if $\triangle$ $a \cdot b \neq 0$ for $a, b \neq 0$, $a, b \in R$ (i.e. all elements regular)

$\triangle$ $I = R x$ (i.e. for every ideal $I \in R$ there is some $x \in R$, s.t. I is generated by x)

Example: $\mathbb{Z}$ is a PID.

Remark: The lemma of Bézout holds in every PID.

Homology computation via the Smith Normal Form

Let R be a PID (e.g. $R = \mathbb{Z}$) and A an $(m \times n)$-matrix with entries in R (e.g. $A = \partial_i$)

Then there is a regular $(m \times m)$-matrix S and a regular $(n \times n)$-matrix T s.t.

$$SAT = \begin{bmatrix} \alpha_1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \ddots & & & & & \\ \vdots & & \ddots & & & & \\ 0 & \cdots & & \alpha_r & \cdots & & 0 \\ 0 & \cdots & & 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \cdots & & 0 & 0 & \cdots & 0 \end{bmatrix},$$

the Smith Normal Form of A.

with $\alpha_i \mid \alpha_{i+1}$ for $i = 1, \dots, r-1$ and $\alpha_i \neq 0$.

Remark: $\triangle$ α_i are called invariant factors of A.

$\triangle$ $\alpha_i = \dfrac{d_i(A)}{d_{i-1}(A)}$ with $\begin{cases} d_i(A) = \gcd \text{ of all } (i \times i)\text{-minus of } A. \\ d_0(A) = 1 \end{cases}$

subdeterminants.

Let

$$D_{i+1} = \begin{bmatrix} \alpha_1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & \alpha_r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

be the Smith Normal Form of $\partial_{i+1} : C_{i+1} \to C_i$ and

$S = \operatorname{rank} C_i - \underbrace{\operatorname{rank} \partial_i}$ can be read off from the SNF of ∂_i.

then

$$H_i(K, \mathbb{Z}) = \mathbb{Z}^{s-r} \oplus \bigoplus_{i=1}^{r} \mathbb{Z}_{\alpha_i}$$

Algorithm for SNF computation

Input: A $(m \times n)$-matrix over R

Output: S, T regular matrices s.t. SAT is a SNF.

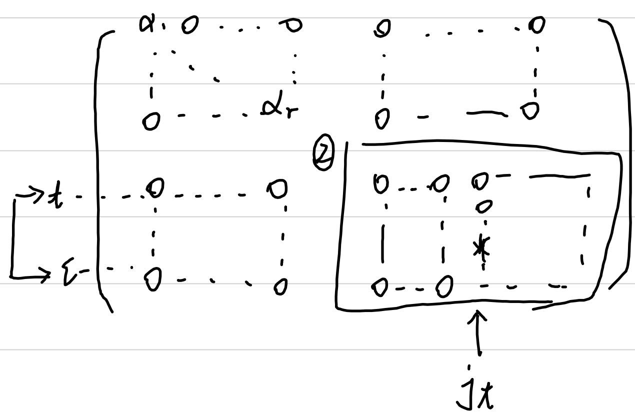
We modify A successively by row and column operations (in mixed order) implemented by regular matrices $S_1, \dots, S_\varepsilon$ and $T_1, \dots, T_e$ s.t.

$$S = S_\varepsilon \cdot S_{\varepsilon-1} \cdots S_2 \cdot S_1.$$

$$T = T_1 \cdot T_2 \cdots T_{e-1} \cdot T_e$$

Initialize: $S = id_m$, $T = id_n$.

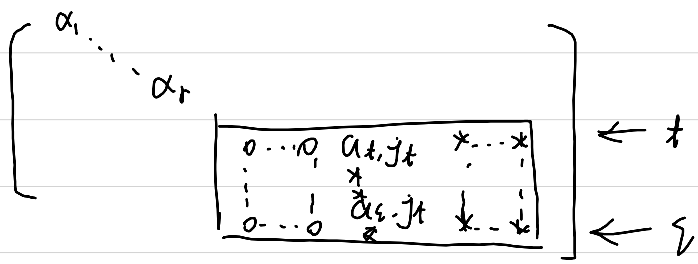
Step I: Proceed recursively and choose pivot



$j_t \geq t$ is the smallest column index with a non-zero entry.

If $a_{t,j_t} = 0$, then there is a smallest ε with $a_{\varepsilon,j_t} \neq 0$ and we exchange the rows $t \leftrightarrow \varepsilon$.

Then



Step II: improve the pivot (should be smallest possible)

If for $a_{t,j_t} \neq 0$ there is an entry $a_{\varepsilon,j_t} \neq 0$ with $a_{t,j_t} \nmid a_{\varepsilon,j_t}$

then let $\gamma_\xi = \gcd(a_{t,j_t}, a_{\varepsilon,j_t})$

By the lemma of Bézout,
there are $\zeta, \tau \in R$
s.t.

$$\gamma_\xi = a_{t,j_t} \cdot \zeta + a_{\varepsilon,j_t} \cdot \tau$$

via row operations (by left multiplication by an invertible matrix S_i)
we replace row t
by $\zeta \times$ row t $+ \tau \times$ row ε.

For $\alpha = a_{t,j_t} / \gamma_\xi$

$\gamma = a_{\varepsilon,j_t} / \gamma_\xi$

we have

$$\zeta \cdot \alpha + \tau \gamma = 1$$

The matrix

$$S_i = \begin{array}{c c} \left[\begin{array}{ccccccc} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \zeta & \cdots & \tau & \\ & & & \vdots & 1 & \vdots & \\ & & & -\gamma & \cdots & \alpha & \\ & & & & & & 1 \\ & & & & & & \ddots \\ & & & & & & 1 \end{array}\right] & \begin{array}{l} \\ \\ \\ \leftarrow t \\ \\ \leftarrow \varepsilon \\ \\ \\ \\ \end{array} \\ \begin{array}{ccccccc} \quad & \quad & \quad & \uparrow & \quad & \uparrow & \\ & & & t & & \varepsilon & \end{array} & \end{array}$$

is then invertible with inverse

$$S_i^{-1} = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \alpha & \cdots & -\tau & \\ & & & \vdots & \ddots & \vdots & \\ & & & \gamma & \cdots & \sigma & \\ & & & & & & 1 \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{bmatrix}$$

and

$$\begin{array}{cl} S_i \cdot A = \begin{bmatrix} \alpha_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \alpha_{r-1} & & & & & \\ & & & 0 & \cdots & 0 & \gamma & * & \cdots & * \\ & & & \vdots & & \vdots & * & \vdots & & \vdots \\ & & & 0 & \cdots & 0 & 0 & * & \cdots & * \\ & & & & & & & & & \end{bmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow t \\ \\ \leftarrow s \\ \\ \end{array} \end{array}$$

We repeat this step until no further improvement is possible.
(since $\gamma \mid a_{t,j_t}$, the procedure terminates)

Step III: eliminate further non-zero entries in row t and column j_t via row and column operations.

$$\begin{pmatrix}
\alpha_1 &&&&&&& \\
& \ddots &&&&&& \\
&& \alpha_{r-1} &&&&& \\
&&& 0 \cdots 0 & \gamma_t & 0 & \cdots & 0 \\
&&& \vdots \quad \vdots & \vdots & * & \cdots & * \\
&&& \vdots \quad \vdots & \vdots & \vdots & & \vdots \\
&&& 0 \cdots 0 & 0 & * & \cdots & *
\end{pmatrix}$$

For row entries: add multiples of row t to remaining rows.

For column entries: repeat step II for columns instead for columns.

Warnings: Column operations may cause new row entries in column j_t and we possibly have to repeat row operations.

However, since γ_t has only finitely many prime factors, the process eventually becomes stationary.

Step IV: Move zero-columns to the right to obtain a matrix

$$\begin{pmatrix}
\alpha_1 &&&&&& \\
& \ddots &&&&& \\
&& \alpha_t &&&& \\
&&&&&& \\
&&&& * & \cdots & * \\
&&&& * & \cdots & * \\
\end{pmatrix}$$

Example: $A = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix}$

$\beta = \gcd(6, 4) = 2$

$\beta = 6 \cdot \sigma + 4 \cdot \tau = 2 \rightarrow \sigma = 1, \ \tau = -1$

$\alpha = 6/2 = 3, \ \gamma = 4/2 = 2$

$\rightarrow \sigma \cdot \alpha + \tau \cdot \gamma = 1 \cdot 3 + (-1) \cdot 2 = 1$

$S_1 = \begin{bmatrix} \sigma & \tau \\ -\gamma & \alpha \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$

T_1
T_2
T_3

$S_1 A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 0 & 7 \end{bmatrix}$.

Add first to second column via $T_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$S_1 A T_1 = \begin{bmatrix} 2 & -2 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \underline{\begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}}$ diag int

Step V: ensure divisibility

If $\alpha_i \nmid \alpha_{i+1}$, add column $i+1$ to column i and apply row operations to 'repair' α_i by replacing it with $\beta = \gcd(\alpha_i, \alpha_{i+1})$

Example: $S_1 A T_1 = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$

Add second to first column via $T_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$(S_1 A T_1) T_2 = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 7 & 7 \end{bmatrix}$$

$$\gamma = \gcd(2, 7) = 1$$

$$\gamma = 2 \cdot \sigma + 7 \cdot \tau = 1 \rightsquigarrow \begin{aligned} \sigma &= -3 \\ \tau &= 1 \end{aligned}$$

$$\alpha = 2/1 = 2$$

$$\beta = 7/1 = 7$$

$$S_2 = \begin{bmatrix} \sigma & \tau \\ -\gamma & \alpha \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -7 & 2 \end{bmatrix}$$

$$S_2 (S_1 A_1 T_1 T_2) = \begin{bmatrix} -3 & 1 \\ -7 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 7 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 0 & 14 \end{bmatrix}$$

Finally, for $T_3 = \begin{bmatrix} 1 & -7 \\ 0 & 1 \end{bmatrix}$

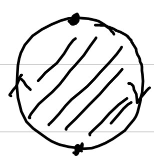
$$(S_2 S_1 A T_1 T_2) T_3 = \begin{bmatrix} 1 & 7 \\ 0 & 14 \end{bmatrix} \begin{bmatrix} 1 & -7 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 14 \end{bmatrix}$$

Smith Normal Form of $A = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix}$

Lecture 2

Example

$\mathbb{R}P^2$ as cell complex (regular CW complex)



1 0-cell (vertex)

1 1-cell (edge)

1 2-cell (disc)

∂_2	2-cell
1-cell	2

∂_1	1-cell
0-cell	0

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

$$\left.\begin{array}{l} Z_2 = \ker \partial_2 = 0 \\ B_2 = \operatorname{im} \partial_3 = 0 \end{array}\right\} H_2 = 0$$

$$\left.\begin{array}{l} Z_0 = \ker \partial_0 = \mathbb{Z} \\ B_0 = \operatorname{im} \partial_1 = 0 \end{array}\right\} H_0 = \mathbb{Z}$$

$$H_1 = \mathbb{Z}^{(\underbrace{\text{rank } C_1}_{1} - \underbrace{\text{rank } \partial_1}_{0}) - 1} \oplus \bigoplus_{i=1}^{1} \mathbb{Z}_2$$

$= \mathbb{Z}_2$ for ∂_2 already in Smith Normal Form.

alternatively $H_1 = \mathbb{Z}/2\mathbb{Z}$

$$Z_1 = \ker \partial_1 = \mathbb{Z}$$
$$B_1 = \operatorname{im} \partial_2 = 2\mathbb{Z}$$

Chapter 14 Persistent Homology

Recall:

Def (filtration)

A filtration of a complex K is a nested of sequence of subcomplexes

$$\emptyset = K^0 \subseteq K^1 \subseteq K^2 \dots \subseteq K^n = K$$

A complex K with a filtration is called a filtered complex.

Examples:

- Čech complex $\check{C}ech_r(S)$
- Vietoris-Rips complex $VR_r(S)$
- Alpha complex $Alpha_r(S) \simeq \check{C}ech_r(S) \cap Voronoi(S)$

- Filtration by dimension

  i.e. $K^l = K_l$, the l-th skeleton.

  with $K^0 = K_{-1} = \emptyset$.

2150 $\emptyset$
2151 $\mathbb{Z}_2$
2152 $\mathbb{Z}_2$
2153 $\mathbb{Z}_{341058}$ ______
2154 $\mathbb{Z}\,\emptyset$

Homology of a filtration

Let K^l be the l-th intermediate complex of a filtration and let

$Z_k^l = Z_k(K^l)$ k-th cycles of K^l

$B_k^l = B_k(K^l)$ k-th boundaries of K^l

$H_k^l = Z_k^l / B_k^l$ k-th homology group of K^l

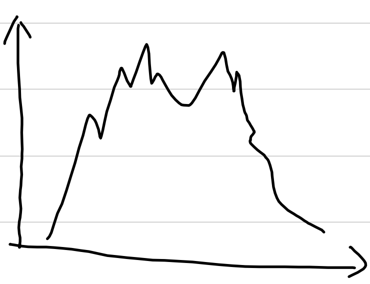
$\beta_k^l = \operatorname{rank}(H_k^l)$ k-th Betti number of K^l.

Over time, β_1^l records the 1-dimensional holes, we call β_1^l a signature functions.

Examples:

S

β_1^l



l

with l depending e.g. on radius r.

Observation:

It is hard to distinguish between

- feature and
- noise

Goal:

We want to identify features as substructures that persist over a longer period of time.

⇝ What are the non-bounding cycles that remain non-boundaries for at least the next p intermediate complexes in the filtration.

Def: (Persistent Homology)

Let (K^l) be a filtration.

Then $H_k^{l,p} = Z_k^l / (B_k^{l+p} \cap Z_k^l)$

is the p-persistent i-th homology group of K^l

$$\beta_k^{l,p} = \text{rank } H_k^{l,p}$$

is the p-persistent i-th Betti number of K^l

Note: $H_k^{l,0} = H_k^l$

Remark: $H_k^{l,p}$ is well-defined because

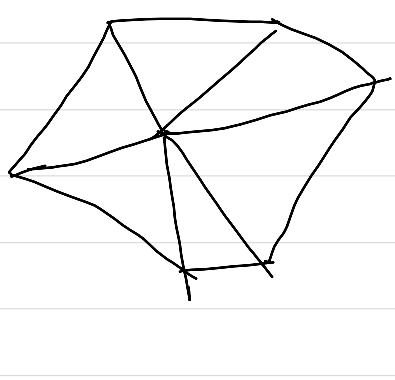
- $B_k^{l+p} \cap Z_k^{l}$ is the intersection of two subgroups of C_k^{l+p} and thus itself a group
- and is a subgroup of Z_k^{l}

Alternatively way to define persistent homology groups:

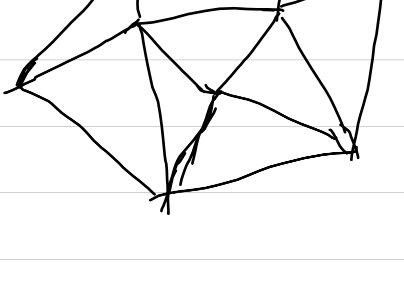
(i) If two cycles are homologous in K^l, then they also exist and are homologous in K^{l+p}
(because $K^l \subseteq K^{l+p}$ as an inclusion gives rise to an injektive simplicial map $K^l \longrightarrow K^{l+p}$

(ii) For the map $\eta_k^{l,p}: H_k^l \to H_k^{l,p}$ that maps a homology class to one containing it, we set

$$H_k^{l,p} = \operatorname{im} \eta_k^{l,p}$$



σ



τ

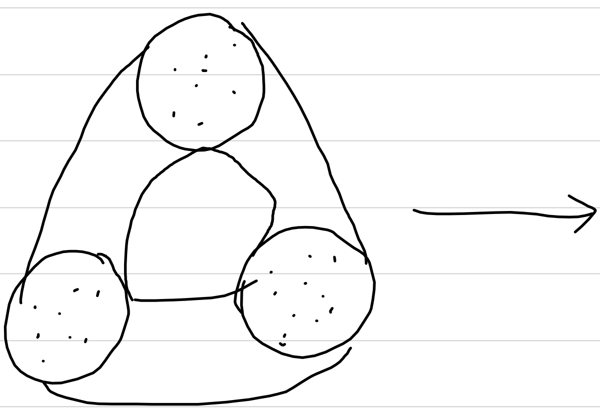
Visualization of persistence.

Suppose (K^1) does not have torsion i.e. choosing $\mathbb{F} = \mathbb{Z}_2$, then for any bounding k-cycle z in the terminal complex K.
then there is a representing pair of simplices (σ, τ) that create and destory $[z]$ at times i, j respectively

Remark: We could have $\tau = \sigma$, though

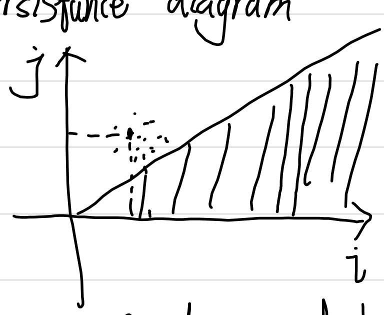
Bar code diagram

Every pair (σ, τ) with $\tau \neq \sigma$ is visualized by a half-open interval $[i, j)$.
the k-interval of the k-cycle z
Non-bounding cycles in K are represented by intervals $[i, \infty)$



This representation of persistence is called hierarchy of bar codes for H_*

Persistance diagram

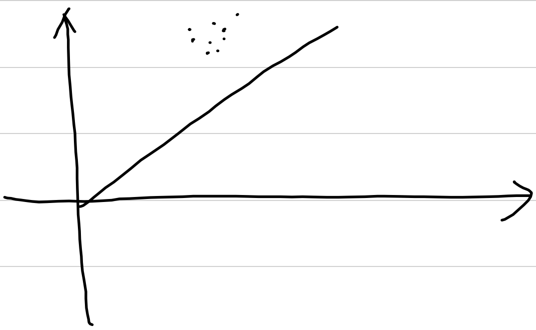


0-dimensional homology
1-dimensional homology

Every point in the persistance diagram represents a pair (σ, τ)
plotted at point (i, j) with
i the birth
j the death
of the pair.

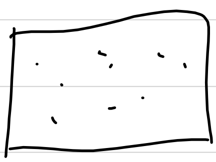
Important:

- All points close to the diagonal (for 1-dimensional or higher homology) (most likely) reprent noise.

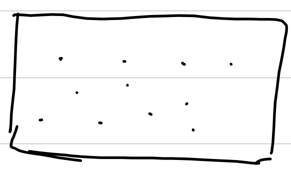


- All points far away from the diagonal represent persistent information of our filtration.

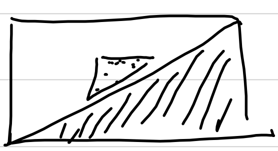
Example: Spin glasses

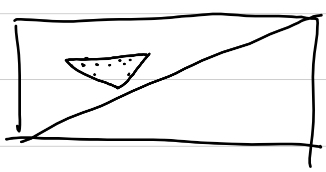


$T = t_1$ (temperature)



$T = t_2$





July. 29 ~ Aug. 8

Aug 25 th ⟶ KTH

Aug 23 | 24